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Fresh orderings of groups

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FRESH ORDERINGS OF GROUPS

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Summary

The notion of a fresh ordering of a group is introduced, and the various closure theorems are proved. The methods involved are as follows: straightforward algebraic techniques, some use of the geometry of trees, and the compactness theorem of first order logic. The notion of a fresh order grows naturally out of the study of linear recurrences in groups. The latter leads to the study of infinite systems of linear equations where the notion of a fresh order proves to be useful.

Chapter 1 introduces the main concepts in the thesis, and starts exploring the class of groups which admit a fresh ordering — fresh groups. Chapter 2 concerns closure properties which the class of fresh groups enjoys. In Chapter 3 we describe a large family of fresh groups and demonstrate that the group ring of a fresh group has no zero-divisors. Chapter 4 concerns proofs by means of mathematical logic, and Chapter 5 is an attempt to modify a result in Chapter 2 by varying the definition of a fresh ordering. In particular an attempt is made to sidestep the earlier use of logical machinery and to give constructive and direct proofs of the result dealt with in the preceding chapters.

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I dedicate this thesis to my parents Elena and Mark Tabachnikov.

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FRESH ORDERINGS OF GROUPS

O. M. Tabachnikova

INTRODUCTION

In recent years, G. C. Smith and two of his research students have investigated recurrence relations in finite $p-groups$.

The doctoral thesis of H. Aydin [1] demonstrated among other things that the fundamental period of a Fibonacci sequence in a finite $p-$group of exponent $p > 3$ and nilpotency class at most 4 is a divisor of the fundamental period of the usual Fibonacci sequence in the integers modulo p. This result can be also found in [3], [4]. The methods used were designed specifically to tackle that particular problem. In the course of proving the main result these methods were used in particular to show that certain quantities vanish modulo p. An example of such a quantity is

$$\sum_{r=0}^{k-1}\sum_{j=0}^{r-1}\sum_{i=0}^{j-1} s_r^2 s_j s_i^2 s_{r-j-1} s_{j-i-1}(-1)^{r+1} + \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} s_j\binom{s_j}{2} s_{j-i-1} s_i^2(-1)^{j+1}$$
$$+\sum_{i=0}^{k-1} s_i^2\binom{s_i}{3}(-1)^{i+1}$$

where k is the fundamental period of the Fibonacci sequence (s_i) modulo p. Here $s_0 = 0$ and $s_1 = 1$.

Aydin and Smith built a range of techniques for showing that such multiple sums vanish, starting with the vanishing of much simpler sums. The calculations in

Aydin's thesis are rather complex and extensive. Therefore we recommend reading the more accessible article [4] of Aydin and Smith. One should note that some elegant group theory is used to reduce significantly the length of the calculation (by studying *Fibonacci automorphisms* of certain relatively free groups) but even so the calculation is still about 40 pages long. Moreover, the methods of the calculations depended crucially on facts about the Fibonacci sequence – for example the *Cassini Identity* $F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}$ (and many more related but more complicated identities). An elementary treatment of these ideas can be found in [5].

The next student of Smith who addressed this area was R. Dikici. Now Dikici and Smith attempted to find a more general approach which would enable them to study (more or less) arbitrary linear recurrences in finite $p-$groups. In other words they wanted to eliminate the dependence on Cassini-like identities. They succeeded in solving this problem by translating it into the language of linear algebra (see [2], [6].) In order to demonstrate that a certain quantity vanishes modulo p they introduce infinitely many new quantities (which in fact are going to vanish modulo p). They find infinitely many (finitary) homogeneous linear equations among these quantities, and then solve this system of linear equations to deduce that all the quantities vanish modulo p, and in particular so does the one quantity of interest.

The way in which the equations were solved is as follows. Select one of the unknowns, and look for a finite but sufficiently large collection of the equations such that any simultaneous solution of these equations must have this unknown vanish. The question arises whether such a technique will necessarily work. Indeed, let us distinguish an unknown; could it be the case that our system of equations admits only the trivial solution, but that any finite subsystem admits a solution where the distinguished unknown fails to vanish? We will demonstrate that, in fact, this cannot happen, and that therefore the technique of Dikici and Smith is bound to succeed, at least in their context. In other words we will demonstrate that the algorithm, which Dikici and Smith developed, of systematically looking at larger and larger finite sets of equations in order to verify that a quantity is 0 modulo p, always terminates. The proof we supply is an application of the Compactness theorem of first order logic due originally to K. Gödel in the countable case; the general case is due to Mal'cev. The history of this theorem is explained in Enderton [7]. The fact that the algorithm of Dikici and Smith terminates is now a consequence of a theorem of mathematical logic – a situation which appears to be novel.

I would like to specifically acknowledge the very helpful conversations and suggestions of Prof C. F. Miller III of the University of Melbourne which contributed to those arguments in this thesis which are based on mathematical logic.

The philosophy of this thesis is to attempt to understand why and how Dikici and Smith succeeded in their quest. In particular, the infinite systems of linear equations which they studied were far from arbitrary, and the methods used for solving them depended on the particular form of these equations. We shall attempt to develop a theory (the *theory of templates*) which underlies their methods. This project immediately leads us to the notion of a *fresh ordering* on a group G. This is a linear ordering of the underlying set of G which satisfies the freshness condition explained below. Note that a fresh ordering does not, in general, make G into an ordered group – a different concept.

Curiously enough, one can prove a local-global theorem for the existence of fresh orderings using the Gödel-Mal'cev compactness theorem. We do this in chapter 4.

In chapter 3 page 73 we will show that if G is a freshly orderable group and R is an integral domain then RG has no zero divisors. This is not subsumed by John Moody's related result on elementary amenable groups which can be found in [13]. Of course, if it were true that every torsion-free group were freshly orderable then this would provide a negative answer to the well-known open question "Can the group ring of a torsion-free group have zero-divisors?" We conjecture that every torsion-free group admits a fresh ordering, but we do not know how one might prove such a result. The conjecture is flimsily based on the absence of counter-examples.

In chapter 2 we investigate the class of freshly orderable groups, and prove that they enjoy various closure properties. We shall also show that a *short-lex* ordering on a free group is necessarily a fresh ordering – the proof differs in spirit from the rest of the thesis because it involves the geometry of the Cayley Graph of the group quite explicitly.

Chapter 1 contains a result which establishes a direct connection between freshly orderable groups and our infinite systems of linear equations. It also gives direct proofs that particular groups are fresh.

Finally, in chapter 5 we shall modify the definition of a fresh ordering and see what theory may be rescued.

Principal Concepts

We shall now explain two principal concepts which are at the heart of this thesis. These are *templates* and *fresh orderings.* We shall also explain the relationship between them.

First of all we formalize the general problem as follows. Given a group G and a field k we consider the set $V = \{f \mid f : G \longrightarrow k\}$ of functions from G to k. If

$U \subseteq G$ then $V_U = \{f \mid f : U \longrightarrow k\}$, thus $V = V_G$.

The collection $\mathfrak{T}$ of all elements of V of finite support is called the *set of templates.* We call T a *system of templates* if T is any subset of $\mathfrak{T}$. Thus

$$T \subseteq \mathfrak{T} = \{f \mid f \in V,\ |\text{supp}(f)| < \infty\}.$$

We define the action of G on V and $\mathfrak{T}$ as follows

$f \cdot g : G \longrightarrow k$

$x \rightarrow (gx)f \ \forall x \in G.$

Now, if $v \in V$ and $t \in \mathfrak{T}$ then we let

$$(v,t) = \sum_{x \in \text{supp}(t)} (x)v \cdot (x)t = \sum_{x \in G} (x)v \cdot (x)t.$$

Here we allow the sum to be infinite because only a finite number of summands are non-zero (since $|\text{supp}(t)| < \infty$).

A template $t \in \mathfrak{T}$ gives rise to a finite linear equation $\sum_{x \in G} y_x \cdot (x)t = 0$ where $\{y_x\}_{x \in G}$ is a collection of unknowns parameterised by the elements of G. A solution to this equation can be viewed as an element of V.

A template $t \in \mathfrak{T}$ gives rise to a system of linear equations via the action of G; these equations are $\sum_{x \in G} y_x(x)(t \cdot g) = 0$ $(\forall g \in G)$ where again $\{y_x\}_{x \in G}$ is a system of unknowns parameterised by the elements of G. We say that these equations are *generated by t on G.*

A solution to the system of equations generated by t on G can be viewed as $v \in V$ such that $(v, t \cdot g) = 0 \ \ \forall g \in G$.

If $T \subseteq \mathfrak{T}$ is a system of templates then we stretch our notation to say that a solution to the system of equations generated by T on G is $v \in V$ such that

$$(v, t \cdot g) = 0 \ \ \forall t \in T,\ \forall g \in G.$$

If $v \in V$ is a solution to the system of equations generated by T on G we will say that the system T of templates *admits a solution v on G.*

We suggest a convenient way of visualising this arrangement via a specific example. Consider the free abelian group G of rank 2. We tile the Euclidean plane by unit squares and distinguish a unit square which will correspond to $1 \in G$. Now G acts in the obvious way on this tiling and each square will correspond to a group element.

A function $v \in V$ is $v : G \longrightarrow k$ and we may imagine the label $(g)v$ as being enscribed in the unit square corresponding to g. Now v can be identified with its set of values, and therefore as some labelling of the tiling. We simply mean that an element of k is written in each tile.

A template $t \in \mathfrak{T}$ has finite support, so it corresponds to a labelling of the tiling where all but finitely many entries are 0.

For illustrative purposes, let $t : G \longrightarrow k$ be such that $(1,0)t = \alpha \neq 0 \neq \beta = (2,0)t$ and $(x,y)t = 0$ otherwise. Thus the support of t has size 2.

A solution of the linear equation associated with t is $v \in V$ such that $(v,t) = 0$. A solution to the system of linear equations generated by t on G is $v \in V$ such that $(v, t \cdot g) = 0 \;\; \forall g \in G$.

As g varies over G, $t \cdot g \in \mathfrak{T}$ is "translated" around $\mathfrak{T}$; the reason for the term template should now be clear. The labelled domino that is associated with t is translated to all possible positions to cover each pair of adjacent horizontal pairs of unit squares by the action of G.

Now we introduce the notion of a fresh ordering of a group. In our example, we are looking for a linear ordering of the elements of G such that at each stage, when considering $t \cdot g$, this template must have some support outside the union of the supports of all preceding $t \cdot h$. Providing our ordering is also a well order, we then have an obvious recursive procedure for building non-zero $v \in V$ such that $(v, t \cdot g) = 0 \;\; \forall g \in G$.

The procedure is as follows: suppose $(s)v$ is defined for all $s \in \bigcup_{h<g} \mathrm{supp}(t \cdot h)$. We can arrange that $(v, t \cdot g) = 0$ by regarding this as a linear equation in at least one unknown (the "unknowns" are $(x)v$ where $x \in \mathrm{supp}(t \cdot g) - \bigcup_{h<g} \mathrm{supp}(t \cdot h)$).

A single non-trivial linear equation in at least two unknowns always has a non-trivial solution. By starting this process in a non-trivial way, we build $v \neq 0, \;\; v \in V$ such that $(v, t \cdot g) = 0 \;\; \forall g \in G$.

Starting the process in a non-trivial way is always possible unless $|\mathrm{supp}(t)| = 1$ in which case t generates the system of linear equations which admits only the trivial solution. Obviously, in the degenerate case if $|\mathrm{supp}(t)| = 0$ then t generates a trivial system of equations which admits any solution $v \in V$. Therefore we will make the following definition.

Definition. *We say that a template $t \in \mathfrak{T}$ is a large template if $|supp(t)| > 1$.*

The procedure we have described makes the following formal definition of a fresh ordering seem natural.

Definition. *A fresh ordering on a group G is a linear ordering $<$ of the underlying set such that whenever P is a non-empty finite subset of G then for each $g \in G$ we have $Pg \not\subseteq \bigcup_{x<g} Px$.*

We recall that an ordering $<$ of a set G is linear if the following properties hold.

1) $\nexists a \in G$ such that $(a < a)$.

2) $(a < b,\ b < c) \Rightarrow (a < c)\ \forall a,\ b,\ c \in G$.

3) $(a < b) \Rightarrow \neg(b < a)\ \forall a,\ b \in G$.

A linear ordering $<$ of a set G becomes a well ordering if any non-empty subset M of G has a minimal element: $\forall M \subset G$ if $M \neq \emptyset$ then $\exists a \in M$ such that $a < b\ \forall b \in M - \{a\}$.

We will also give a list of related definitions including that of a well fresh ordering.

We can modify our picture of the group $G = \mathbb{Z}^2$ acting on a tiled plane. Let $\widehat{G}$ be the additive group of the integers $\mathbb{Z}$. $\widehat{G}$ acts on $\mathbb{R}$, the Euclidean line, in an obvious way. We may consider the integer z as being a label for the interval $[z - \frac{1}{2},\ z + \frac{1}{2})$, and proceed as in the case of $\mathbb{Z}^2$. In this case solving the system of linear equations generated by the template $\hat{t}$ being a restriction of t to $\widehat{G}$ can be easily done using the technique mentioned above.

We suppose for illustrative purposes that the support P of $\hat{t}$ consists of two adjacent squares. The coefficients associated with the squares of P are α and β. We start with placing P over the group elements g_1 and $g_1 + 1$ on the "squared line" to obtain the equation $x_{g_1} \cdot \alpha + x_{g_1+1} \cdot \beta = 0$. We can solve it by assigning an arbitrary non-zero value to x_{g_1} and calculating x_{g_1+1} from the equation.

Next we move P gradually to the right one square at a time. This way at every step we cover one new square or in other words we consider an equation with exactly one new variable. The value of the other variable has been calculated at the previous step. After having done this indefinitely to the right we return to the original position and start backtracking to the left in the same fashion.

This illustrates building a non-trivial solution of the system of linear equations generated by a single large template $\hat{t}$ on $\widehat{G} = \mathbb{Z}$ using a suitable ordering $\sqsubset$ of the group $\mathbb{Z}$. We can use the ordering

$$0 \sqsubset -1 \sqsubset -2 \sqsubset -3 \sqsubset \ldots \sqsubset 1 \sqsubset 2 \sqsubset \ldots .$$

However, there are many orderings of $\mathbb{Z}$ which would have been equally suitable. For example, the orderings

$$1 \sqsubset 0 \sqsubset -1 \sqsubset -2 \sqsubset \ldots \sqsubset 2 \sqsubset 3 \sqsubset 4 \sqsubset \ldots$$

or

$$1 \sqsubset 2 \sqsubset 0 \sqsubset 3 \sqsubset -1 \sqsubset 4 \sqsubset -2 \sqsubset 5 \sqsubset -3 \sqsubset \ldots .$$

We will see from the following list of definitions and considerations of chapter 1 that they are fresh orderings of $G = \mathbb{Z}$. The set of all fresh orderings of $\mathbb{Z}$ is in fact infinite and is fully described in chapter 1.

Definition. *A group is said to be fresh or freshly orderable if there exists a fresh order of G.*

Definition. *A well fresh ordering on a group G is a fresh ordering on G which happens to be a well ordering of the underlying set.*

Definition. *A group is said to be well fresh or well freshly orderable if there exists a well fresh order of G.*

Definition A. *A group is said to be locally fresh or locally freshly orderable if whenever P is a non-empty finite subset of G there exists a linear order $<_P$ of G such that for each $g \in G$ we have $Pg \not\subseteq \bigcup_{x <_P g} Px$. The order $<_P$ is said to be P-fresh on G (or fresh for P on G.)*

Definition B. *A group is said to be locally well fresh or locally well freshly orderable if whenever P is a non-empty finite subset of G there exists a well order $<_P$ of G such that for each $g \in G$ we have $Pg \not\subseteq \bigcup_{x <_P g} Px$. The order $<_P$ is said to be well P-fresh on G (or well fresh for P on G.)*

It is proved in chapter 2 that the closure properties which freshly orderable groups enjoy as a class include being closed under formation of arbitrary Cartesian products, free products and extensions. We also observe that they are closed under taking subgroups.

We also demonstrate in chapter 3 that left (or right) orderable groups are freshly orderable. The study of left (or right) orderable groups can be found in the book by Mura and Rhemtulla [14].

We note that the system of numbering the results of this thesis has the following features. The first digit of a number of a result refers to the number of the chapter where this result is first stated. Propositions, statements, corollaries and lemmas have numbers according to those of the theorems to which they are related.

CHAPTER 1

In this chapter we present an important result which was mentioned in the introduction. This result is stated below as theorem 1.1. It establishes a direct connection between the problem of solving systems of linear equations with the notion of a well fresh ordering of a group. Next we will give some examples of the groups which admit well fresh orderings. First we will remind the reader of our notation.

G is a group with a well fresh order $<$, and k is a field,

$V = \{f \mid f : G \longrightarrow k\}$ is the set of functions from G to k,

If $U \subseteq G$ then $V_U = \{f \mid f : U \longrightarrow k\}$, thus $V = V_G$.

$\mathfrak{T} = \{f \mid f \in V, |\text{supp}(f)| < \infty\}$ is the set of all "templates". A *system of templates* is any subset of $\mathfrak{T}$. Recall that a template t is large if $|\text{supp}(t)| > 1$.

G acts on V and $\mathfrak{T}$ as follows:

$f \cdot g : G \longrightarrow k$

$x \mapsto (gx)f \quad \forall x \in G.$

Thus V is a right kG-module and $\mathfrak{T}$ is a submodule.

If $v \in V$ and $t \in \mathfrak{T}$ then

$$(v, t) = \sum_{x \in \text{supp}(t)} (x)v \cdot (x)t = \sum_{x \in G} (x)v \cdot (x)t,$$

where we allow the sum to be infinite since only a finite number of summands are non-zero (because $|\text{supp}(t)| < \infty$).

Now we state the theorem itself.

Theorem 1.1. *Let G be a non-trivial group with a well fresh order $<$. Let a system of templates T consist of a single large template t. It follows that $\exists v \in V - \{0\}$ such* *that* $(v, t \cdot g) = 0 \; \forall g \in G$.

Aside: In other words, if G is a well freshly orderable non-trivial group then the infinite system of linear equations generated by one large template t on G has a non-trivial solution.

(Remark: The system of equations is infinite since G is infinite. This follows from G being fresh, because as we will soon see (p.17), fresh groups are torsion-free.)

Proof. We shall call a set $I \subseteq G$ an initial segment of G if $I \neq \emptyset$ and if $(g \in I,\ h < g) \Rightarrow (h \in I)$.

Let S be the set of all initial segments of G. We note that $G \in S$. We will show that S is a well ordered set with respect to inclusion.

Indeed, if $I \neq G$ is a proper initial segment we define its successor

$$\mathrm{succ}(I) = \min_{<}\{a \mid a \in G,\ a \notin I\}.$$

Now we can write I as follows

$$I = \{b \mid b \in G,\ b < \mathrm{succ}(I)\}.$$

It is now clear that $(I_1 = I_2) \Leftrightarrow (\mathrm{succ}(I_1) = \mathrm{succ}(I_2))$ where $I_1,\ I_2 \in S$. Also $(I_1 \subseteq I_2) \Leftrightarrow (\mathrm{succ}(I_1) \leq \mathrm{succ}(I_2))$ from which it follows that the order $\subset$ is linear.

Now we will show that $\subset$ is also a well order. We assume for contradiction that it is not so. This means that there is an infinite descending chain of elements of S :

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \dots$$

This is equivalent to the existence of the infinite descending chain of successors of these sets:

$$\mathrm{succ}(I_1) > \mathrm{succ}(I_2) > \dots$$

which contradicts $<$ being a well order of G. Thus $\subset$ on S is indeed a well order.

Next we define a cover of I : $C(I) = \bigcup_{g \in I} \operatorname{supp}(t \cdot g)$. These are all those elements of the group G which will get "covered" by the support of our template t when we move $\operatorname{supp}(t)$ by acting on t by the elements of the segment I.

We define $\Omega = \{(I, \alpha) \mid I \in S,\ \alpha \in V_{C(I)},\ \alpha \neq 0\}$. In other words these are all possible pairs of initial segments and non-trivial functions from the covers of these segments to the field k. (So $\alpha : \bigcup_{g \in I} \operatorname{supp}(t \cdot g) \longrightarrow k$, $\exists y \in \operatorname{dom}(\alpha)$ such that $(y)\alpha \neq 0$).

We will impose a partial order $\sqsubseteq$ on the set Ω as follows.

$$((I_1, \alpha_1) \sqsubseteq (I_2, \alpha_2)) \Leftrightarrow ((I_1 \subseteq I_2),\ (\alpha_2|_{C(I_1)} = \alpha_1))$$

In other words I_2 contains I_1 and α_2 is an extension of α_1 from $C(I_1)$ to $C(I_2)$ such that they coincide on $C(I_1)$.

We modify our definition of $(\ ,\ )$ into $(\ ,\ )_{C(I)}$ since we wish to give meaning to $(\alpha, t \cdot g)$ where $g \in I$. We let

$$(\alpha, t \cdot g)_{C(I)} = \sum_{x \in C(I)} (x)\alpha \cdot (x)(t \cdot g).$$

We define $\Gamma \subseteq \Omega$ as follows

$$\Gamma = \{(I, \alpha) \mid (I, \alpha) \in \Omega,\ (\alpha,\ t \cdot g)_{C(I)} = 0\ \forall g \in I\}.$$

In other words these are such pairs where the first element is an initial segment and the second element is, as it were, an approximation to a solution of the system of linear equations generated by the template t on G.

Next we are going to show that $\Gamma \neq \emptyset$ and that Zorn's lemma applies to the set Γ. Then we will be able to conclude that Γ has a maximal element $(J, \tilde{v})$. Finally we will show that in fact $J = G$ and therefore $\tilde{v}$ is the required non-trivial solution of our infinite system of linear equations on G.

We will implement this plan in stages. First of all we show that $\Gamma \neq \emptyset$. We denote by g_1 the minimal element of the group G with respect to the order $<$ (such an element exists since $<$ is a well order). We define a function v on $\mathrm{supp}(t \cdot g_1)$ such that $(v, t \cdot g_1)_{C(\{g_1\})} = 0$ and $v \neq 0$ (v is not a zero-function). Since $|\mathrm{supp}(t)| > 1$ it follows that $|\mathrm{supp}(t \cdot g_1)| > 1$. We calculate v from the equation

$$0 = (v, t \cdot g_1)_{C(\{g_1\})} = \sum_{x \in \mathrm{supp}(t \cdot g_1)} (x)v \cdot (x)(t \cdot g_1)$$

by assigning to $(x)v$ arbitrary values (but not all zeros) from k for all $x \in \mathrm{supp}(t \cdot g_1)$ except one $\hat{x} \in \mathrm{supp}(t \cdot g_1)$ (remember that $|\mathrm{supp}(t \cdot g_1)| \geq 2$). We find $(\hat{x})v$ from the equation above:

$$(\hat{x})v = \frac{- \sum\limits_{x \in (\mathrm{supp}(t \cdot g_1) - \hat{x})} (x)v(x)(t \cdot g_1)}{(\hat{x})(t \cdot g_1)},$$

where $(\hat{x})(t \cdot g_1) \neq 0$ since $\hat{x} \in \mathrm{supp}(t \cdot g_1)$. We now let $I = \{g_1\}$ and so we have $C(I) = \mathrm{supp}(t \cdot g_1)$. Therefore we obtain $(I, v) \in \Gamma$. Thus, indeed, $\Gamma \neq \emptyset$.

Next we will show that Zorn's lemma applies to Γ. In other words we will show that any linearly ordered subset of the partially ordered set Γ has an upper bound.

Let L be a linearly ordered non-empty subset of Γ. So, $\emptyset \neq L \subseteq \Gamma$ and L is linearly ordered.

Let $A = \bigcup_{(I,\alpha) \in L} I$. It follows from the definition of an initial segment that A is also an initial segment.

Let β be a function β: $C(A) \longrightarrow k$ defined as follows. We note that $\forall c \in C(A)$ $\exists (I,\ \alpha) \in L$ such that $c \in C(I)$ (since $A = \bigcup_{(I,\alpha) \in L} I$). Then $(c)\beta = (c)\alpha$.

We will now check that β is well defined. We assume that $c \in C(I_1) \cap C(I_2)$ where (I_1, α_1), $(I_2, \alpha_2) \in L$. Without loss of generality we assume that $I_1 \subseteq I_2$. Then $\alpha_2|_{C(I_1)} = \alpha_1$ and hence $(c)\alpha_1 = (c)\alpha_2$. Thus β is well defined.

We will now show that $(A,\ \beta) \in \Gamma$ or, in other words, that $(\beta, t \cdot g)_{C(A)} = 0\ \forall g \in A$. But this is obvious since if $g \in A$ then $g \in I$ for some $(I, \alpha) \in L$ and so $(\beta,\ t \cdot g)_{C(I)} = 0$ by the definition of β. Also $\beta \neq 0$ (is not a zero-function) since no α is a zero-function. Finally, from the design of A and β it follows that $(I,\ \alpha) \sqsubseteq (A,\ \beta)$ for each $(I, \alpha) \in L$ so that $(A,\ \beta)$ is an upper bound of L.

Thus Zorn's lemma applies to Γ which means that there is a maximal element $(J,\ \tilde{v})$ in Γ. It remains to show that $J = G$ so that $\tilde{v}$ is a non-trivial solution of our system of linear equations on G.

We assume for contradiction that it is not so, that is to say $J \neq G$. Thus $G - J \neq \emptyset$ and so there is the minimal element g_{min} of the set $G - J$ with respect to the well order $<$ of G. We note that $J \bigcup \{g_{\text{min}}\} = \bar{J} \supsetneq J$.

Next we will define a function $\bar{\alpha}$ on $C(\bar{J})$ so that

$$\bar{\alpha}|_{C(J)} = \tilde{v}$$

and

$$(\bar{\alpha}, t \cdot g_{\text{min}})_{C(\bar{J})} = 0.$$

To this end we have in fact to repeat the same process as the one we used when defining a function v on $\text{supp}(t \cdot g_1)$ in order to show that $\Gamma \neq \emptyset$.

We know that $(\star)$: $\text{supp}(t \cdot g_{\text{min}}) \not\subseteq C(J)$ since $<$ is a fresh order of G. We want the following to hold

$$\sum_{x \in \operatorname{supp}(t \cdot g_{\min})} (x)\bar{\alpha}(x)(t \cdot g_{\min}) = 0. \tag{$\star\star$}$$

$\bar{\alpha}$ is already defined on $C(J)$. Thus we have to extend $\bar{\alpha}$ to $\operatorname{supp}(t \cdot g_{\min})$. So we have to define $\bar{\alpha}$ for $x \in (\operatorname{supp}(t \cdot g_{\min}) - C(J))$ so that $(\star\star)$ holds.

To this end we assign arbitrary values from k to $(x)\bar{\alpha}$ for all $x \in (\operatorname{supp}(t \cdot g_{\min}) - C(J))$ except one $\hat{x}$. This time we do not have to worry about making sure that these values are not all zeros since $(x)\bar{\alpha} \neq 0$ on $C(J)$ and so $\bar{\alpha}$, which we are building on a larger set, will not be a zero-function.

Next we calculate $(\hat{x})\bar{\alpha}$ from the equation $(\star\star)$. We note that if $(\operatorname{supp}(t \cdot g_{\min}) - C(J))$, which is not empty because of $(\star)$, consists of just one x then this x will be our $\hat{x}$ and we will have nothing to assign.

We make the calculations as follows:

$$(\hat{x})\bar{\alpha} = \frac{-\displaystyle\sum_{x \in \operatorname{supp}(t \cdot g_{\min}) - \{\hat{x}\}} (x)\bar{\alpha}(x)(t \cdot g_{\min})}{(\hat{x})(t \cdot g_{\min})}.$$

Thus we have built the required function $\bar{\alpha}$ on $C(\bar{J})$ and so we have an element $(\bar{J},\ \bar{\alpha})$ from Γ which by design is bigger than the element $(J,\ \tilde{v})$. This violates the maximality of the latter. The contradiction proves that $J \neq G$ is false and thus $J = G$. Thus we have found an element $(G,\ \tilde{v}) \in \Gamma$ which means that $\tilde{v}$ is a required non-trivial solution of the infinite system of linear equations generated by the large template t on G. Theorem 1.1 is now proved.

We note that for the purposes of studying infinite systems of linear equations the property of being locally well fresh is just as important as that of being well fresh.

Indeed, theorem 1.1 ensures the existence of a non-trivial solution of the system of linear equations generated by a large template t if our group G is well fresh. However, if G is locally well fresh then there exists an ordering which is fresh for the set P where $P = \mathrm{supp}(t)$. We then use this ordering exactly as in the proof of theorem 1.1 to build a non-trivial solution to the system of linear equations generated by t.

Moving on, the first quite straight forward but important feature of fresh groups is that they must be torsion-free and therefore infinite, if not trivial.

Indeed, if a group $G \neq 1$ is not torsion-free then it can not be fresh for the following reason. Let $g \in G$, $g \neq 1$ be an element of finite order n in G. We consider the set $P = \{1, g, \dots, g^{n-1}\}$. Now, if G were fresh then we would have a fresh order $<$ on G. With respect to the order $<$ either $1 < g$ or $g < 1$. We suppose that $1 < g$. We note that $P \cdot 1 = P \cdot g = P$ since $Pg = \{g, g^2, \dots, g^n = 1\} = P$. We can now write $Pg \subseteq \bigcup_{x<g} Px$. This is because $Pg = Px = P$ when $x = 1 < g$. If $g < 1$ a similar argument works.

Thus the assumption that a group G with non-trivial torsion is fresh led us to a contradiction. Therefore every fresh group is torsion-free.

Going back to theorem 1.1 we note the condition of G being fresh is crucial in the sense that without it the theorem is no longer true.

Indeed, if we drop the restriction on G being fresh then generally speaking the conclusion of theorem 1.1 is no longer true. This means that for a group G which is not necessarily fresh a single large template may admit a non-trivial solution in certain cases and may not in others. Before we demonstrate this we note that the class of groups with torsion is a supply of groups which are definitely not fresh.

Statement 1.1.1. *Suppose G is a group and that t is a large template such that the sum of its values vanishes. The equations generated by t on G admit a non-trivial solution u, where u is any non-zero constant function on G.*

Statement 1.1.2. *Let G be a group with torsion and k be a field. There is a large template t on G which admits only the trivial solution on G if any one of the conditions below holds:*

a) $|k| > 3$.

b) $|k| = 3$ *and there is* $a \in G$ *such that* $2 < o(a) < \infty$.

c) $|k| = 2$ *and there is* $a \in G$ *such that* $3 < o(a) < \infty$.

We will now prove both statements.

Proof of statement 1.1.1. Let t be a large template such that the sum of its coefficients is zero. In other words let $t: G \longrightarrow k$, $1 < |\text{supp}(t)| < \infty$, $\sum_{x \in \text{supp}(t)} (x)t = 0$. Let u be a constant non-zero function on G. That is to say $u = c$ on G, where $c \in k$, $c \neq 0$. We have $\forall g \in G \quad \sum_{gh \in \text{supp}(t)} (h)u \cdot (gh)t = c \cdot \sum_{x \in \text{supp}(t)} (x)t = 0$. This means that a non-zero constant function u on G is indeed a solution on G of the system of linear equations generated by t.

To prove statement 1.1.2 we will need the following auxiliary propositions.

Lemma 1.1.2 (1). *If $C =< a >$ is a finite non-trivial cyclic group then given a field k there is a large template t which admits only the trivial solution on C (unless $|C| = 2,\ |k| \leq 3$ or $|C| = 3,\ |k| = 2$ in which case non-trivial solutions exist for such templates.)*

Proof of lemma 1.1.2 (1). Let $|C| = n$. First we consider the case $n > 2$, $|k| \neq 2$.

Let $t : C \longrightarrow k$ be the template such that $(a^{n-1})t = 1 - \alpha$ and $(g)t = 1$ if $g \neq a^{n-1}$. Choose $\alpha \notin \{0,\ n\}$ which is possible since $|k| \neq 2$. Suppose $u : C \longrightarrow k$ is such that $(u, t \cdot g) = 0\ \forall g \in C$. Let $(a^i)u = u_i\ \forall i \in \{1, \ldots, n\}$ then $\sum_{i=1}^{n} u_i \cdot (a^{i+j})t = 0\ \forall j \in \{0, \ldots, n-1\}$. This will force $u = 0$ if and only if the following matrix is non-singular[1]:

$$\begin{pmatrix} 1 & 1 & 1 & . & . & 1 & 1-\alpha \\ 1 & 1 & 1 & . & . & 1-\alpha & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1-\alpha & 1 & 1 & . & . & 1 & 1 \end{pmatrix}. \qquad (\star)$$

Let $W = k^n$ be a vector space over the field k of dimension n, where n is the cardinality of the group C which we have assumed to be larger than 2. Next, we define the following set of vectors in W.

$$\{e_i = (0, 0, 0, \ldots, 0, 1, 0, \ldots, 0), i \in \{1, \ldots, n\}\},$$

where 1 is in the ith position. We let $w = (1, 1, 1, \ldots, 1) \in W$. We also let $w_i = w - \alpha e_i\ \ \forall i \in \{1, \ldots, n\}$.

We can regard the rows of our matrix as being vectors of W. Clearly, the sequence of these vectors coincides with the sequence $(w_i)_{i=1}^{n}$. So the matrix $(\star)$ is non-singular if and only if the sequence $(w_i)_{i=1}^{n}$ is a linearly independent sequence of vectors in W.

Suppose $\sum_{i=1}^{n} \lambda_i w_i = 0$ then $\sum_{i=1}^{n} \lambda_i (w - \alpha e_i) = 0$. Now we put $\sum_{i=1}^{n} \lambda_i = r \in k$, then $rw = \alpha \sum_{i=1}^{n} \lambda_i e_i$ so $r = \alpha \lambda_i\ \forall i \in \{1, \ldots, n\}$. Now we sum over i to obtain $nr = \alpha r$ and so $(n - \alpha)r = 0$.

Now $\alpha \neq \{0,\ n\}$ so $n - \alpha \neq 0$ and thus $r = 0$. We conclude that $\alpha \lambda_i = 0\ \forall i \in \{1, \ldots, n\}$ but $\alpha \neq 0$ so $\lambda_i = 0\ \forall i \in \{1, \ldots, n\}$ and we are done.

[1]The external examiner, Dr A. Camina, has pointed out a shorter and more elegant proof of the non-singularity of this matix.

Now we focus on the case $|k| = 2$. We can re-use the above argument using $\alpha = 1$ providing n is even and so $n = 0$ in k. Thus we may assume n is odd. Suppose $n > 3$ and consider the template $s : C \longrightarrow k$ where $(a^{n-1})s = (a^{n-2})s = 0$ and $(a^i)s = 1$ if $0 \leq i < n-2$. Now s is a large template since $n > 3$ and we seek to show that if $u : C \longrightarrow k$ and $(u, s \cdot g) = 0 \;\forall g \in C$ then $u = 0$. It suffices to show that the matrix

$$\begin{pmatrix} 1 & 1 & 1 & . & . & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & . & . & 1 & 0 & 0 & 1 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 1 & 1 & . & . & 1 & 1 & 1 & 0 \end{pmatrix} \qquad (\star\star)$$

is non-singular. Let $\widehat{w_i} = w - e_i - e_{i+1}$ (subscript taken modulo n.) Suppose $\sum_{i=1}^{n} \lambda_i \widehat{w_i} = 0$ then $(\sum_{i=1}^{n} \lambda_i)w = \sum_{i=1}^{n} \lambda_i e_i + \sum_{i=1}^{n} \lambda_i e_{i+1}$. Let $v = \sum_{i=1}^{n} \lambda_i$ then equating coordinates we obtain $v = \lambda_i + \lambda_{i-1} \;\forall i \in \{1, \ldots, n\}$. Thus $nv = \sum_{i=1}^{n} \lambda_i + \sum_{i=2}^{n+1} \lambda_{i-1} = 2v$ and therefore $(n-2)v = 0$. Now, n is odd, so $n - 2 = 1$ in k so $v = 0$ and hence $\lambda_i = -\lambda_{i-1} = \lambda_{i-1} \;\forall i \in \{1, \ldots, n\}$. Now we can introduce $\lambda = \lambda_i \;\forall i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^{n} \lambda_i \widehat{w_i} = \lambda \sum_{i=1}^{n} \widehat{w_i} = \lambda w$ since n is odd. Now, $0 = \lambda w$ so $\lambda = 0$ and so $\lambda_i = 0 \;\forall i \in \{1, \ldots, n\}$. Thus the matrix $(\star\star)$ is non-singular.

It remains to study the case $n = 3, |k| = 2$. The only large template which might admit only the trivial solution gives rise to the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix}$$

where only one of α, β or γ is allowed to be 0 and the others are forced to be 1. Without loss of generality we may assume that $\gamma = 0$ and $\alpha = \beta = 1$. Thus we have a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

which is obviously singular. Therefore we conclude that for $|C| = 3$, $|k| = 2$ any large template admits a non-trivial solution.

Finally we consider the case $n = |C| = 2$. In this case we take t such that $(1)t = c$, $(a)t = d$, where $c \neq \pm d$. This condition can always be satisfied as long as the field k has order $|k| > 3$. We have a system

$$c \cdot (1)u + d \cdot (a)u = 0$$

$$d \cdot (1)u + c \cdot (a)u = 0$$

where u is a solution. The determinant of this system is $c^2 - d^2 \neq 0$ since $c \neq \pm d$. So $(1)u = (a)u = 0$ is a unique solution.

However, if $|k| \leq 3$ and $|C| = 2$ (a case singled out in the hypothesis of the lemma) then again any large template is forced to be: $t : C \longrightarrow k$ such that

$$(1)t = c \neq 0$$

$$(a)t = d \neq 0$$

since $|\mathrm{supp}(t)| \geq 2 = |C|$. Thus we have a system

$$c \cdot (1)u + d \cdot (a)u = 0$$

$$d \cdot (1)u + c \cdot (a)u = 0.$$

The determinant $c^2 - d^2$ of this system is equal to zero because necessarily $c = \pm d$. If it were not so we would have either three distinct non-zero elements in the field of size at most 3, or two distinct non-zero elements where one is equal to its negation. This is clearly absurd. Therefore in this case there exists a non-trivial solution.

Thus the lemma is proved for all non-trivial finite cyclic groups.

Lemma 1.1.2 (2). *Let G be a group which is not torsion-free. Let $H \leq G$ be a finite subgroup such that there is a large template t on H which admits only the* *trivial solution on H, then there is a large template t' on G which admits only the trivial solution on G.*

Proof of lemma 1.1.2 (2). Given $t : H \longrightarrow k$ a template on H we extend it to $t' : G \longrightarrow k$ by putting $(g)t' = 0$ for $g \notin H$. Let u be a solution on G which this extended template t' admits. Now, u has the property that

$$(\sum_{a\in H}(a)u(ba)t = 0 \ \forall b \in H) \Rightarrow ((a)u = 0 \quad \forall a \in H)$$

Now we know that

$$\sum_{c\in G}(c)u \cdot (dc)t' = 0 \ \forall d \in G.$$

So

$$\sum_{c\in G}(c)u \cdot (bdc)t' = 0 \ \forall d \in G \ \forall b \in H.$$

Now as c ranges over G so does $a = dc$. Therefore

$$\sum_{a\in G}(d^{-1}a)u \cdot (ba)t' = 0 \ \forall b \in H, \ \forall d \in G.$$

Now, $\text{supp}(t') \subseteq H$ and $b \in H$, so

$$\sum_{a\in H}(d^{-1}a)u \cdot (ba)t = 0 \quad \forall b \in H, \ \forall d \in G,$$

or rather

$$\sum_{a\in H}(a)(u \cdot d^{-1}) \cdot (ba)t = 0 \quad \forall b \in H, \ \forall d \in G.$$

Thus $(a)(u \cdot d^{-1}) = 0 \ \forall a \in H \quad \forall d^{-1} \in G$. So, $(d^{-1}a)u = 0 \quad \forall a \in H \quad \forall d^{-1} \in G$, that is to say $(g)u = 0 \quad \forall g \in G$.

Given these two lemmas a proof of statement 1.1.2 is now obvious.

Proof of statement 1.1.2. Since G is not torsion-free then there is an element $a \in G$, $a \neq 1$ such that $o(a) < \infty$. Thus $< a >$ is finite and non-trivial.

If k is such that $|k| > 3$ then such a choice of a is sufficient for our purposes. If, however, k is such that $|k| \leq 3$ we choose a special $a \in G$ depending on the size of k. For $|k| = 3$ we select a such that $o(a) > 2$. For $|k| = 2$ we select a such that $o(a) > 3$. The existence of such a is ensured by the hypothesis of the statement.

Thus we have either $| < a > | > 2$, $|k| = 3$ or $| < a > | > 3$, $|k| = 2$ or our original case $1 < | < a > | < \infty$, $|k| > 3$. By lemma 1.1.2 (1) in any of these cases there is a large template t which admits only the trivial solution on $< a >$. Now let $H = < a >$ and apply lemma 1.1.2 (2). We obtain that there is a large template t' on G admitting only the trivial solution on G.

Thus we have shown that the requirement in theorem 1.1 of G being fresh is indeed crucial.

Next we will give some examples of well freshly orderable groups.

We start with the group $\mathbb{Z}$ of integers. The following result is a criterion to determine if a linear ordering on $\mathbb{Z}$ is fresh.

Theorem 1.2. *A linear ordering $\sqsubset$ of $\mathbb{Z}$ is fresh if and only if for all $z \in \mathbb{Z}$, the following condition holds: Whenever a, b, $c \in \mathbb{Z}$ with $a \leq b \leq c$ (natural ordering) and $a \sqsubset z$ and $c \sqsubset z$ then $b \sqsubset z$.*

Proof. We first assume that the above condition holds. We will show that our linear ordering $\sqsubset$ is then fresh.

Suppose P is a non-empty finite subset of $\mathbb{Z}$. Let $p_{\min}$ and $p_{\max}$ be the smallest and the largest elements of P with respect to the natural ordering of $\mathbb{Z}$. Suppose (for contradiction) that there exists an integer z such that

$$P + z \subseteq \bigcup_{x \sqsubset z} P + x.$$

Thus $p_{\min} + z,\ p_{\max} + z \in \bigcup_{x \sqsubset z} P + x$. Thus there exist elements $x_1,\ x_2 \in \mathbb{Z}$ and $p_1,\ p_2 \in P$ with $x_1,\ x_2 \sqsubset z$ such that $p_{\min} + z = p_1 + x_1$ and $p_{\max} + z = p_2 + x_2$. Thus

$$x_1 \leq x_1 + (p_1 - p_{\min}) = z = x_2 + (p_2 - p_{\max}) \leq x_2.$$

We now use the condition in the statement of this lemma to deduce that $z \sqsubset z$ which is absurd. Thus $\sqsubset$ is indeed fresh.

Conversely, we will now assume that a linear ordering $\sqsubset$ is fresh. We will show that the above condition then holds.

Suppose for contradiction that the condition does not hold. This means that there are $a,\ b,\ c \in \mathbb{Z}$ such that $a \leq b \leq c,\ a \sqsubset z,\ c \sqsubset z$ but $z \sqsubseteq b$ for some $z \in \mathbb{Z}$. Now we will build a non-empty subset P of $\mathbb{Z}$ such that for some $g \in \mathbb{Z}$ the ordering $\sqsubset$ will fail to be fresh. In other words the inclusion $P + g \subseteq \bigcup_{x \sqsubset g} P + x$ will hold.

We construct P to have the following properties:

$$(1) \quad p_{\max} - p_{\min} = c - a,$$

where $p_{\max}$ and $p_{\min}$ are maximal and minimal elements of P with respect to the natural ordering on $\mathbb{Z}$.

$$(2) \quad p \in P \Leftrightarrow p_{\min} \leq p \leq p_{\max}$$

which means that P is "connected".

Next, let $g = b$. We will now show that such P and g indeed form a counter-example to the freshness of $\sqsubset$.

We have for all $p \in P$

$$p_{\min} + a \le p_{\min} + b \le p + b \le p_{\max} + b.$$

Now one of the following two cases is possible

$$(a) \quad p + b \le p_{\max} + a$$

or

$$(b) \quad p_{\max} + a \le p + b.$$

In the case (a) we have

$$p_{\min} + a \le p + b \le p_{\max} + a$$

which means that

$$(p + b) \in (P + a) \subseteq \bigcup_{x \sqsubset b} (P + x)$$

since $a \sqsubset b$ (because $a \sqsubset z,\ z \sqsubseteq b$) and also since P is connected according to the property (2).

In the case (b) we have $p_{\min} + c = p_{\max} + a \le p + b \le p_{\max} + b \le p_{\max} + c$. The first equality here follows from the property (1) of P. In other words we have

$$p_{\min} + c \le p + b \le p_{\max} + c$$

which means that

$$(p + b) \in (P + c) \subseteq \bigcup_{x \sqsubset b} (P + x)$$

since P is connected (property (2) of P) and since $c \sqsubset b$ (because $c \sqsubset z,\ z \sqsubseteq b$). Thus, since $p \in P$ is arbitrary, and either case (a) or case (b) applies, we have

$$(P + b) \subseteq \bigcup_{x \sqsubset b} (P + x).$$

This contradicts the ordering $\sqsubset$ being fresh on $\mathbb{Z}$. Thus our assumption was wrong and so the condition holds.

The theorem is now proved.

By having this criterion we in fact have an algorithm for building a well fresh ordering $\sqsubset$ on $\mathbb{Z}$. We describe it inductively as follows.

1) Pick up an arbitrary integer a_1.

2) Choose the next integer a_2 in the ordering to be either $a+1$ or $a-1$. Thus we have built $X(2) = (a_1,\ a_2);\ a_1 \sqsubset a_2$.

3) Given a sequence of adjacent (with respect to $\sqsubset$) integers $X(n) = (a_1, \dots, a_n)$ such that $a_1 \sqsubset a_2 \sqsubset \dots a_n$ choose the next integer a_{n+1} in the ordering to be either $(\max_< X(n)) + 1$ or $(\min_< X(n)) - 1$. Here $\max_<$ and $\min_<$ means respectively maximum and minimum with respect to the natural order $<$ of integers. So we built $X(n+1) = (a_1, \dots, a_{n+1}),\ a_1 \sqsubset \dots \sqsubset a_{n+1}$.

Thus for every $i \in \mathbb{N}$ we have described a collection $X(i)$ of the first i elements ordered by $\sqsubset$. We note that $\forall i,\ j$ such that $i < j$ we have $X(j)$ extends $X(i)$. Now we can form the sequence X which is, informally, $\lim_{i\to\infty} X(i)$. We now identify the sequence X with its set of elements.

If $X = \mathbb{Z}$ then we have already ordered $\mathbb{Z}$. Otherwise, if $X \neq \mathbb{Z}$ it is easy to see that we have one of the following two cases.

(a) $\exists a \in \mathbb{Z}$ such that $X = \{x \mid x \geq a\}$ or (b) $\exists b \in \mathbb{Z}$ such that $X = \{x \mid x \leq b\}$.

Now we have to order $\mathbb{Z} - X$. If we are in the case (a) we order $\mathbb{Z} - X$ as follows: $a-1 \sqsubset a-2 \sqsubset \dots$ and for any $x \in X,\ y \in \mathbb{Z} - X\ \ x \sqsubset y$.

Similarly, in the case (b) we order $\mathbb{Z} - X$ as follows: $b+1 \sqsubset b+2 \sqsubset \dots$ and for any $x \in X,\ y \in \mathbb{Z} - X\ \ x \sqsubset y$.

We claim that the ordering $\sqsubset$ built in this way is a well fresh ordering.

Indeed, if we have $a \leq b \leq c$ in the natural ordering $<$ of $\mathbb{Z}$ and $a \sqsubset z,\ c \sqsubset z$ (for an arbitrary $z \in \mathbb{Z}$) in the built ordering $\sqsubset$ then $b \sqsubset z$. Otherwise, if $z \sqsubseteq b$ we have either $a \sqsubset c \sqsubset z \sqsubseteq b$ or $c \sqsubset a \sqsubset z \sqsubseteq b$. In other words either $a \sqsubset c \sqsubset b$ or $c \sqsubset a \sqsubset b$. This means that $b < \min_{<}\{a,\ c\}$ or $b > \max_{<}\{a,\ c\}$ which is to say that either $b < a$ or $b > c$. This is absurd since $a \leq b \leq c$.

Thus $\sqsubset$ is a fresh ordering on $\mathbb{Z}$. It is also a well ordering.

Conversely, any well fresh ordering of $\mathbb{Z}$ can be built using the above algorithm.

Indeed, if we have a well fresh ordering $\sqsubset$ then pick its minimal element a_1 and then the minimal element a_2 of $\mathbb{Z} - a_1$ with respect to $\sqsubset$. So we have $a_1 \sqsubset a_2$ and $a_2 \sqsubset b$ for any $b \in \mathbb{Z},\ b \neq a_1,\ a_2$.

We want to prove that either $a_2 = a_1 + 1$ or $a_1 - 1$. If it is not so then there is $b \in \mathbb{Z}$ such that $a_1 < b < a_2$ or $a_2 < b < a_1$. But $a_1 \sqsubset a_2 \sqsubset b$ so we have $a_1 \sqsubset b,\ a_2 \sqsubset b,\ a_1 < b < a_2$ (or $a_2 < b < a_1$.) Now, if $a_1 < b < a_2$ ($a_2 < b < a_1$) then $a_1 \leq b \leq a_2$ ($a_2 \leq b \leq a_1$.) Thus we can apply theorem 1.2 to conclude that $b \sqsubset b$ which is absurd.

Next, having a sequence of adjacent (with respect to $\sqsubset$) integers $X(n+1) = (a_1, \dots, a_{n+1}),\ a_1 \sqsubset \dots \sqsubset a_n \sqsubset a_{n+1}$ we assume for contradiction that $a_{n+1} \neq (\max_{<} X(n)) + 1$ and $a_{n+1} \neq (\min_{<} X(n)) - 1$ where $X(n) = X(n+1) - a_{n+1}$. There is $b \in \mathbb{Z}$ such that either $b = (\max_{<} X(n)) + 1$ or $b = (\min_{<} X(n)) - 1$. So $b \notin X(n)$. We therefore have either $(\max_{<} X(n)) < b < a_{n+1}$ or $a_{n+1} < b < (\min_{<} X(n))$. Also, since $\max_{<} X(n) = a_i$ $(1 \leq i \leq n)$ and $\min_{<} X(n) = a_j$ $(1 \leq j \leq n)$ and $b \notin X(n)$ we have $a_i \sqsubset b,\ a_j \sqsubset b,\ a_{n+1} \sqsubset b$. Thus we have the following. Either

$a_i < b < a_{n+1}$, $a_i \sqsubset b$, $a_{n+1} \sqsubset b$ or $a_{n+1} < b < a_j$, $a_j \sqsubset b$, $a_{n+1} \sqsubset b$. Applying theorem 1.2 in both cases we have $b \sqsubset b$ which is absurd.

Thus we have a description of the set S of all well fresh orderings of $\mathbb{Z}$ - they are all those which can be built using the above algorithm. We can use this description to establish a cardinality of this set S.

We observe that well fresh orderings with the minimal element a are in bijective correspondence with infinite binary sequences. Indeed, given $a \in \mathbb{Z}$ we can "read" any infinite binary sequence consisting of 0 and 1 as a set of rules for building a well fresh ordering $\sqsubset$ on $\mathbb{Z}$ with a as the minimal element. Since when building a well fresh ordering according to our algorithm we have two options at every step we will interpret each of the two possible entries (0 or 1) of our infinite binary sequence as an order which option to take at every step. We do it as follows.

First, we denote a by a_1. Next, if a sequence starts with 0 we make $a_2 = a_1 - 1$. Otherwise, if it starts with 1, we make $a_2 = a_1 + 1$. Also, given $X(n) = (a_1, \dots, a_n)$, $a_1 \sqsubset a_2 \sqsubset \dots \sqsubset a_n$ we look at the element of the sequence in the position n. If it is 0 we make $a_{n+1} = (\min_< X(n)) - 1$. Otherwise, if it is 1, we make $a_{n+1} = (\max_< X(n)) + 1$. If our sequence has only finitely many zeros (or finitely many 1) then we will end up having built $X \subset \mathbb{Z}$ where X is infinite and is of the form $X = \{x \mid x \geq d\}$ (or $X = \{x \mid x \leq b\}$) for some $d \in \mathbb{Z}$ ($b \in \mathbb{Z}$). We complete X to $\mathbb{Z}$ by making $d-1 \sqsubset d-2 \sqsubset \dots$ (or $b+1 \sqsubset b+2 \sqsubset \dots$) and bigger than any element of X.

Thus given any $a \in \mathbb{Z}$ there is a bijection between the set of all well fresh orderings of $\mathbb{Z}$ with a as the minimal element and all infinite binary sequences.

On the other hand there is an obvious bijection between the set of infinite binary sequences and the set T of subsets of $\mathbb{N}: \{s_i\}_{i=1}^{\infty} \mapsto \{i \mid s_i = 1\}$.

The set T of subsets of $\mathbb{N}$ has the cardinality of the continuum c. Therefore the set of all well fresh orderings of $\mathbb{Z}$ with a as the minimal element being in bijective correspondence with T also has the cardinality c.

The minimal element a can range over a countable set $\mathbb{Z}$. Therefore the set S of all distinct well fresh orderings of the integers has the cardinality of the continuum c.

It follows now that $\mathbb{Z}$ is a well freshly orderable group where S described above is the set of all its well fresh orderings. However, the natural ordering $<$ on $\mathbb{Z}$, though is fresh since it satisfies the condition of theorem 1.2, is not in S since it is not a well ordering. It turns out that the set S together with just two more linear orderings of $\mathbb{Z}$ which are fresh but not well exhaust all fresh orderings of the integers. These two orderings are the natural ordering $<$ of the integers and the one opposite to it.

Indeed, suppose we have integers in their natural order and a fresh ordering $\sqsubset$ on $\mathbb{Z}$. Between any two adjacent integers b and $b+1$ we insert the symbol $\sqsubset$ if $b \sqsubset b+1$ and the symbol $\sqsupset$ if $b+1 \sqsubset b$.

Now, if for any b we have $b \sqsubset b+1 \sqsupset b+2$ then we have the violation to a freshness condition since $b \leq b+1 \leq b+2$, $b \sqsubset b+1$, $b+2 \sqsubset b+1$ implies $b+1 \sqsubset b+1$ which is absurd. Thus this situation cannot occur.

However, if we have $b \sqsupset b+1 \sqsubset b+2$ this simply means that $b+1$ is the minimal element of a fresh ordering $\sqsubset$ and so, according to what we proved before, $\sqsubset$ is a well fresh ordering from S. Indeed, if $b+1$ were not the minimal element of $\mathbb{Z}$ with respect to $\sqsubset$ then it would force the change of the symbols $\sqsubset$ or $\sqsupset$ into the opposite ones to happen somewhere in the chain of the integers we are looking at. We have eliminated this possibility.

Finally, the only remaining possibility is that the opposite symbols $\sqsubset$ and $\sqsupset$ do not meet. In other words neither of the cases $b \sqsubset b+1 \sqsupset b+2$; $b \sqsupset b+1 \sqsubset b+2$ ever occurs. But this means that either for all $b \in \mathbb{Z}$ we have $b \sqsubset b+1$ or for all $b \in \mathbb{Z}$ we have $b \sqsupset b+1$. In other words we either have $\sqsubset$ being the same ordering as the natural ordering $<$ of the integers or the opposite to it and we are done.

Next, we use theorem 1.2 to prove that a group $\mathbb{Z}^n$ is also well fresh, although this result will be subsumed by theorem 2.1 of chapter 2.

Theorem 1.3. *The free abelian group $\mathbb{Z}^n$ is well freshly orderable.*

Proof. Let $<$ be the natural ordering on $\mathbb{Z}$. Let $<_{\mathbb{Z}}$ be a well fresh ordering of $\mathbb{Z}$. If $x, y \in \mathbb{Z}^n$ we write $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Let $\sqsubset$ be the following ordering of $\mathbb{Z}^n : x \sqsubset y$ exactly when the following conditions hold.

$$\exists j \ \ (1 \le j \le n) \text{ such that } x_j <_{\mathbb{Z}} y_j \text{ and}$$

$$x_i = y_i \ \forall i < j.$$

Clearly, it is a well ordering of $\mathbb{Z}^n$. We claim that $\sqsubset$ is also fresh on $\mathbb{Z}^n$. Let us suppose for contradiction that it is not so. Then $\exists P \subset \mathbb{Z}^n$, $P \neq \emptyset, |P| < \infty$ and $\exists y \in \mathbb{Z}^n$ such that

$$P + y \subset \bigcup_{x \sqsubset y} P + x \qquad (\star).$$

We will write $x \sqsubset^j y$ if $x_i = y_i \ \forall i < j$ and $x_j <_{\mathbb{Z}} y_j$. We put $X_j = \{x \mid x \sqsubset^j y\}$.

Now, either $\forall x \in X_j \ x_j < y_j$ or $\forall x \in X_j \ x_j > y_j$. If it were not so then we would have $\exists \tilde{x}, \ \hat{x} \in X_j$ such that $\tilde{x}_j \le y_j \le \hat{x}_j$ and $\tilde{x}_j <_{\mathbb{Z}} y_j, \ \hat{x}_j <_{\mathbb{Z}} y_j$. By theorem 1.2 we would then have $y_j <_{\mathbb{Z}} y_j$ which is absurd.

Now we will choose $\hat{p} \in P$ such that it will give rise to the necessary contradiction.

Let us build $\hat{p}$ inductively.

If $y_1 > (<) x_1$ for all $x \in X_1$ then $\hat{p}_1 = \max(\min)\{p_1\}$ where $p \in P$.

Let $P_1 = \{p \in P \mid p_1 = \hat{p}_1\}$.

If $y_i > (<) x_i$ for all $x \in X_i$ then $\hat{p}_i = \max(\min)\{p_i | p \in P_{i-1}\}$.

Let $P_i = \{p \in P \mid p_k = \hat{p}_k,\ k \leq i\}$.

Now for $\hat{p},\ y\ \exists \tilde{p},\ \tilde{x}$ by $(\star)$ such that

$$\hat{p} + y = \tilde{p} + \tilde{x}, \quad \tilde{x} \sqsubset y.$$

$\tilde{x} \sqsubset y$ means $\exists j$ such that $\tilde{x} \sqsubset^j y$. That is

$$\hat{p}_k - \tilde{p}_k = \tilde{x}_k - y_k = 0\ \forall k < j$$

$$\hat{p}_j - \tilde{p}_j = \tilde{x}_j - y_j \neq 0.$$

So $\tilde{p} \in P_{j-1}$.

If $\tilde{x}_j - y_j < 0$ then we have

$$\hat{p}_j - \tilde{p}_j = \tilde{x}_j - y_j < 0$$

and by definition of $\hat{p}_j$ we have $\hat{p}_j > \tilde{p}_j$. So

$$0 \leq \hat{p}_j - \tilde{p}_j = \tilde{x}_j - y_j < 0$$

which is absurd.

Similarly, if $\tilde{x}_j - y_j > 0$ then we obtain

$$0 \geq \hat{p}_j - \tilde{p}_j = \tilde{x}_j - y_j > 0$$

which is absurd as well.

The theorem is now proved.

Thus we have shown that free abelian groups of finite rank are examples of well freshly orderable groups and proved that on such groups a large template always admits a non-trivial solution.

CHAPTER 2

In this chapter we study the property of being freshly orderable in relation to group theoretic constructions. We show that the class of fresh groups enjoys various closure properties. We then use this knowledge to prove certain families of groups to be those of fresh groups.

First we will generalise the result of theorem 1.3 of chapter 1 by proving that the property of a group being fresh is inherited by forming not just finite but arbitrary Cartesian products. However, we can only show that the property of being well fresh is preserved by forming finite Cartesian products.

Before stating the relevant result we introduce the following standard notation. We wish to think of a Cartesian product of groups as a collection of functions rather than as a collection of "vectors".

Let $(G_i)_{i\in I}$ be a family of groups indexed by the set I. The Cartesian product of these groups has underlying set

$$\mathrm{Cart}_{i\in I}G_i = \{f \mid f : I \longrightarrow \bigcup_i G_i,\ (i)f \in G_i\ \forall i \in I\}.$$

The group law is given pointwise. If $f_1, f_2 \in \mathrm{Cart}_{i\in I}G_i$ then

$$f_1f_2 : I \longrightarrow \bigcup_i G_i,\ (i)(f_1f_2) = (i)f_1(i)f_2\ \forall i \in I.$$

We have natural epimorphisms $\pi_i : \mathrm{Cart}_{i\in I}G_i \longrightarrow G_i$ given by $(f)\pi_i = (i)f$.

Theorem 2.1. *Let $(G_i)_{i\in I}$ be a family of freshly orderable groups and suppose that $G = Cart_{i\in I}G_i$. It follows that the group G is freshly orderable.*

Proof 1. We invoke the well ordering principle to obtain a well order $<_I$ on the indexing set I. We suppose that each group G_i has a fresh ordering $<_i$. We know that G comes equipped with projection epimorphisms π_i. We define a relation $<^j$ on G as follows. If $g, h \in G$ and $j \in I$ we write $g <^j h$ if $(j)g <_j (j)h$ but $(i)g = (i)h \ \forall i <_I j$.

We write $g < h$ if and only if there exists $j \in I$ such that $(j)g <^j (j)h$. It is easy to see that $<$ is an ordering of G. In fact $<$ is a linear ordering. This is because if $g \neq h$ then $\Omega = \{i \mid i \in I, (i)g \neq (i)h\}$ is a non-empty subset of the well ordered set I. Let j be the minimal element of Ω and we are done.

Suppose that P is a non-empty subset of G and that this P in conjunction with $g \in G$ gives rise to a counter-example to our proposition (for contradiction). Thus $Pg \subseteq \bigcup_{x<g} Px$. Fixing P and g, for each $j \in I$ we define certain subsets of G by

$$A_j = \{f \mid f \in G, f <^j g\}$$

and

$$B_j = \bigcup_{i <_I j} PA_i.$$

The set P is finite so there exist $y_1, \ldots y_n \in G$ with $y_1 < y_2 < \ldots < y_n < g$ and $Pg \subseteq \bigcup_{1 \leq_{\mathbb{N}} \alpha \leq_{\mathbb{N}} n} Py_\alpha$, where $\leq_{\mathbb{N}}$ denotes the natural order of $\mathbb{N}$. Among all such finite sequences choose one such that $k \in I$ is minimized (with respect to $<_I$) where $y_n <^k g$. Notice that k exists since $<_I$ is a well ordering.

Now we have $y_n < g$ and $Pg \subseteq \bigcup_{y \leq y_n} Py$. Let $g_k \in G$ be such that $(i)g_k = 1$ if $i <_I k$ but $(i)g_k = (i)g$ if $i \geq_I k$. Let $T = (Pg - B_k)g_k^{-1}$.

Notice that B_k is invariant under right multiplication by g_k^{-1}. In fact, it is invariant under right multiplication by any $h \in G$ such that $(j)h = 1 \ \forall j <_I k$. Indeed, $B_k = \bigcup_{i <_I k} PA_i$. So, if $x \in B_k$ then $x \in PA_j$ for some $j <_I k$ and so $x \in P\hat{x}$ where $\hat{x} <^j g$ and $j <_I k$. This means $(i)\hat{x} = (i)g, \ i <_I j, \ (j)\hat{x} <_j (j)g$.

Now $(j)(\hat{x}h^{-1}) = (j)\hat{x}(j)h^{-1} = (j)\hat{x} \cdot 1$ since $(j)h^{-1} = (j)h = 1$ for $j <_I k$ and therefore $(j)(\hat{x}h^{-1}) = (j)\hat{x} <_j (j)g$.

Also, for $i <_I j$ $(i)(\hat{x}h^{-1}) = (i)\hat{x}(i)h^{-1} = (i)\hat{x} \cdot 1 = (i)g$.

Thus $\hat{x}h^{-1} <^j g$ $(j <_I k)$ and so $\hat{x}h^{-1} \in A_j$. This means that $xh^{-1} \in P\hat{x}h^{-1} \subset PA_j \subset B_k$. Thus if $x \in B_k$ then $xh^{-1} \in B_k$ and so $x \in B_k h$. Therefore $B_k \subset B_k h$.

A similar argument works to show that the converse inclusion $B_k h \subset B_k$ is true for all h such that $(j)h = 1$ $\forall j <_I k$ and therefore $B_k = B_k h$. In particular, $B_k = B_k g_k^{-1}$. Going back to our set T we can now conclude that $T = Pgg_k^{-1} - B_k$. Also observe that $(i)gg_k^{-1} = (i)g$ if $i <_I k$ but $(i)gg_k^{-1} = 1$ if $i \geq_I k$.

This set T will give rise to the necessary contradiction. First observe that $T \neq \emptyset$ since otherwise we would have $Pg \subseteq B_k = \bigcup_{i<_I k} PA_i$. Using the finiteness of Pg we could then choose $w_1, \ldots w_m$ in $\bigcup_{i<_I k} A_i$ such that $Pg \subseteq \bigcup_{1 \leq_{\mathrm{N}} \beta \leq_{\mathrm{N}} m} Pw_\beta$. So for any β such that $1 \leq_{\mathrm{N}} \beta \leq_{\mathrm{N}} m$ we have $w_\beta \in A_i$ for some $i <_I k$. This means that $\exists w_\beta \in G,\ w_\beta <^i g$, where $i <_I k$. This is absurd since it contradicts the minimality of k.

Thus T is non-empty.

Now

$$Tg_k = (Pg - B_k) \subseteq \bigcup_{x \leq y_n} (Px - B_k) \subseteq \bigcup_{x\leq y_n,\ x<^t g,\ t \geq_I k} (Pgg_k^{-1}g_k g^{-1}x - B_k).$$

Here we inserted $gg_k^{-1}g_k g^{-1} = 1$. Also notice that the exclusion of $x <^t g$ for $t <_I k$ is harmless because of the definition of B_k. Now

$$\{g_k g^{-1}x \mid x \leq y_n, x <^t g, t \geq_I k\} \subseteq \Lambda,$$

where

$$\Lambda = \{z \mid z \in G, (i)z = 1 \text{ if } i <_I k,\ (k)z \leq_k (k)y_n\}.$$

Indeed, if $i <_I k$ then $(i)x = (i)g$, $(i)g_k = 1$, so $(i)z = (i)(g_k g^{-1}x) = 1 \cdot (i)g^{-1}(i)g = 1$. If $i = k$ then $(k)x \leq_k (k)y_n$, $(k)g_k = (k)g$, so $(k)z = (k)(g_k g^{-1}x) = (k)g_k(k)g_k^{-1}(k)x = (k)x \leq_k (k)y_n$. So

$$Tg_k \subseteq \bigcup_{z\in\Lambda}(Pgg_k^{-1}z - B_k) = \bigcup_{z\in\Lambda}(Pgg_k^{-1} - B_k)z = \bigcup_{z\in\Lambda} Tz.$$

Here we used the fact that $B_k = B_k z$, because clearly $(i)z = 1$ if $i <_I k$.

Now apply the projection $\pi_k : G \longrightarrow G_k, g \mapsto \bar{g}$ to the inclusion $Tg_k \subseteq \bigcup_{z\in\Lambda} Tz$ to obtain that

$$\bar{T}\bar{g}_k \subseteq \bigcup_{z\in\Lambda} \bar{T}\bar{z}.$$

Notice that $(k)y_n <_k (k)g_k = \bar{g}_k$. Thus

$$\bar{T}\bar{g}_k \subseteq \bigcup_{\bar{z}\in G_k,\ \bar{z}<_k\bar{g}_k} \bar{T}\bar{z}.$$

Here we may have increased the set over which z ranges by relaxing the condition on z. Indeed, if $z \in \Lambda$ then $\bar{z} \in G_k$ (since $z \in G$) and $\bar{z} <_k \bar{g}_k$ (since $\bar{z} = (k)z \leq_k (k)y_n <_k (k)g_k = \bar{g}_k$). So $(z \in \Lambda) \Rightarrow (\bar{z} \in G_k,\ \bar{z} <_k \bar{g}_k)$.

The last inclusion contradicts the hypothesis that the relation $<_k$ on G_k is a fresh ordering.

We will give an alternative proof of the theorem. To this end we introduce two lemmas which are also used to prove several other results.

Lemma 2.1.1. *Let $<$ be a linear ordering of a group G which is not fresh. Let $\Omega = \{P \mid P \subset G,\ P \neq \emptyset,\ |P| < \infty,$ such that $<$ is not fresh for $P\}$.*

Let $P \in \Omega$ be such that $|P|$ is of minimal size among elements of Ω.

Let $g \in G$ be such that $Pg \subseteq \bigcup_{h<g} Ph$.

Then the following statements are true.

a)For any $p \in P$ there exists $q \in P - \{p\}$ such that $qg = ph$ for some $h < g$. Moreover, if $p' \in P$ and $h' \in G$ with $h' < g$ are such that $qg = p'h'$ then $p = p'$ and $h = h'$.

b)There are elements $h_i \in G,\ h_i < g,\ i \in \{1, \dots, n\}$ such that the elements of P can be arranged in the following way:

$$p_2 g = p_1 h_1$$

$$\dots$$

$$p_{i+1} g = p_i h_i$$

$$\dots$$

$$p_1 g = p_{n+1} g = p_n h_n$$

where $p_i \neq p_j$ if $i \neq j$ for $(i, j \in 1, \dots, n),\ p_{n+1} = p_1$.

Proof. First of all $\Omega \neq \emptyset$ since $<$ is not fresh on G. For the same reason the set of $g \in G$ such that $Pg \subseteq \bigcup_{h<g} Ph$ is not empty. It is also clear that $|P| \geq 2$ since any linear ordering on G is fresh for a singleton set (recall definition A given in the introduction (p. 9) of a linear ordering on a group being fresh for a set). Otherwise we would have $g \in \bigcup_{h \in G,\ h<g} \{h\}$ that is to say $g < g$ which is absurd. We will now prove parts a) and b) of the lemma in sequence.

a)Let $Q = P - p \neq \emptyset$. By minimality of the size of P the ordering $<$ is fresh for Q.

Thus $Qg \not\subseteq \bigcup_{h \in G,\ h<g} Qh$ but $Pg \subseteq \bigcup_{h \in G,\ h<g} Ph$.

Choose $q \in Q$ such that $qg \not\subseteq \bigcup_{h\in G,\ h<g} Qh$ then $qg \in Pg$ so

$$qg \in (\bigcup_{h\in G,\ h<g} Ph) - (\bigcup_{h\in G,\ h<g} Qh).$$

Therefore $qg = ph$ for some $h < g$.

Moreover, if $p' \in P$ and $h' \in G,\ h' < g$ are such that $qg = p'h'$ then $p = p'$ and so $h = h'$.

b)Choose any $p \in P$ and let $p_1 = p$. Now suppose p_i has been chosen and select $p_{i+1} \in P$ as follows. Let p_i be the element p in lemma 2.1.1 part a). Let p_{i+1} be an element $q \in P$ satisfying the condition of lemma 2.1.1 part a), which means that $p_{i+1}g = p_ih_i$ with $h_i < g$. Since P is finite the infinite sequence (p_i) must have a first repeated entry. By omitting initial terms and relabelling we may assume that $p_1, p_2, ..., p_m$ are distinct but $p_{m+1} = p_1$.

Now $p_{i+1}g = p_ih_i$ for some $h_i < g\ \forall i < m,\ i \geq 1$. Also $p_1g = p_{m+1}g = p_mh_m$. Thus $\{p_1, ..., p_m\} \subseteq P$ and $\{p_1, ..., p_m\} \in \Omega$ since $\{p_1, ..., p_m\}g \subseteq \bigcup_{h<g}\{p_1, ..., p_m\}h$ which means that $<$ is not fresh for $\{p_1, ..., p_m\}$. The minimality of $n = |P|$ forces $n = m$ and $P = \{p_1, p_2, ..., p_n\}$. Thus we have rearranged the points of P in such a way that the equations above now hold.

Lemma 2.1.2. *Let G be a group. Let $<$ be a linear ordering on G. Let $P = \{p_1, \dots, p_n\} \subset G$ where $p_1, \dots, p_n$ (not necessarily different) are such that the following system of equations holds*

$$p_2 g = p_1 h_1$$

$$\dots$$

$$p_{i+1} g = p_i h_i \qquad (\bigwedge)$$

$$\dots$$

$$p_1 g = p_{n+1} g = p_n h_n$$

where $h_i \leq g \ \forall i \in \{1, \dots, n\}$ *and* $\exists j \in \{1, \dots, n\}$ *such that* $h_j < g$.

Then $<$ *is not fresh on* G.

Proof. The proof is by induction on n.

Such a system of equations can only arise if $n \geq 2$. Suppose $n = 2$ and, without loss of generality, suppose $h_1 < g$. Thus $p_2 \neq p_1$ but $p_1 g = p_2 h_2$, so $h_2 \neq g$. Thus $\{p_1, p_2\} g \subseteq \bigcup_{h<g} \{p_1, p_2\} h$ and so $<$ is not a fresh ordering.

Now suppose $n > 2$. If $h_i < g$ for all i then we have $Pg \subseteq \bigcup_{h<g} Ph$ and so $<$ is not a fresh order.

Thus we may assume $h_i = g$ for some i; relabelling, if necessary, we may suppose $h_n = g$.

Now

$$p_1 g = p_n h_n = p_n g = p_{n-1} h_{n-1}.$$

Thus we can omit the penultimate equation of $(\bigwedge)$ and replace the final equation by $p_1 g = p_{n-1} h_{n-1}$. We now have a smaller system of equations satisfying the hypothesis of the lemma. This is because otherwise the equation $p_n g = p_{n-1} h_{n-1}$ which we removed had to be the only equation with $h_{n-1} < g$. But in this case we would have $h_i = g \forall i \in \{1, \dots, n\},\ i \neq n-1$. This forces $p_1 = \dots = p_{n-1} = p_n$ which together with $p_n g = p_{n-1} h_{n-1}$ implies that $g = h_{n-1} < g$. This is absurd. Thus, by induction, $<$ is not a fresh order of G.

The lemma is now proved.

We can now present an alternative proof to theorem 2.1.

Proof 2. First we invoke the same ordering $<$ on G as in the first proof. We will prove that $<$ is fresh on G.

Let us assume for contradiction that it is not so. We select $P \subseteq G$ a finite non-empty set of minimal size such that there exists $g \in G$ with $Pg \subseteq \bigcup_{h<g} Ph$. We may apply lemma 2.1.1. We obtain a system of equations

$$p_2 g = p_1 h_1$$

$$\dots$$

$$p_{i+1} g = p_i h_i \tag{$\star$}$$

$$\dots$$

$$p_1 g = p_{n+1} g = p_n h_n$$

where $h_i < g\ (i \in \{1, \dots, n\}),\ p_i \neq p_j$ if $i \neq j\ (i, j \in \{1, \dots, n\}), p_{n+1} = p_1$. From the definition of the ordering $<$ of G it follows that given $g \in G$ we can associate with every $h \in G,\ h < g$ a unique element $\alpha_h \in I$ such that $(\alpha_h)h <_{\alpha_h} (\alpha_h)g$ and $\forall \beta <_I \alpha_h\ \ (\beta)h = (\beta)g$

Now let $\alpha = \min\{\alpha_{h_1}, \dots, \alpha_{h_n}\}$ where the minimum is taken with respect to the ordering $<_I$.

Thus we have found $\alpha \in I$ such that (1): $\exists j \in \{1, \dots, n\}$ with the property $(\alpha)h_j <_\alpha (\alpha)g$ and (2): $\forall \beta <_I \alpha \;\; (\beta)h_i = (\beta)g \; \forall i \in \{1, \dots, n\}$ (this is because $(\beta <_I \alpha \leq_I \alpha_{h_i} \; \forall i \in \{1, \dots, n\}) \Rightarrow ((\beta)h_i = (\beta)g))$.

Next we apply the projection $\pi_\alpha : G \longrightarrow G_\alpha, \; g \mapsto \bar{g}$ to the system $(\star)$ to get

$$\bar{p}_2 \bar{g} = \bar{p}_1 \bar{h}_1$$

$$\dots$$

$$\bar{p}_{i+1} \bar{g} = \bar{p}_i \bar{h}_i \qquad (\star\star)$$

$$\dots$$

$$\bar{p}_1 \bar{g} = \bar{p}_{n+1} \bar{g} = \bar{p}_n \bar{h}_n$$

where $h_i < g$ $(i \in \{1, \dots, n\})$ results in $\bar{h}_i \leq_\alpha \bar{g} \; \forall i \in \{1, \dots, n\}, \; i \neq j$ together with $\bar{h}_j \leq_\alpha \bar{g}$ because of the statements (2) and (1) respectively. Also, $\bar{p}_1, \dots, \bar{p}_n$ are not necessarily all different since π_α is not necessarily injective.

Now we apply lemma 2.1.2 using G_α as the group G, $(P)\pi_\alpha$ as the set P, $<_\alpha$ as the order $<$ and $(\star\star)$ as the system $(\bigwedge)$ of equations in the lemma respectively.

We conclude that $<_\alpha$ is not fresh on G_α which is a contradiction.

The proof is now completed.

However, if we are interested in a more specific property of groups — that of being well fresh rather than just fresh — then we will need the following proposition.

First we set up our notations. Let $(G_i)_{i \in I}$ be a family of non-trivial well orderable groups. Let $G = \mathrm{Cart}_{i \in I} G_i$. We define the ordering $<_G$ on G as follows. For $g, h \in G$ we write $g <_G h$ if $\exists j \in I$ such that $(j)g <_j (j)h$ and $(i)g = (i)h \; \forall i <_I j$, where $<_j$ is a well order on G_i and $<_I$ is a well order on I.

Proposition 2.1. *In this notation the order $<_G$ is a well order on the group G if and only if the indexing set I is finite.*

Proof. If I is finite then $I = \{1, 2, \ldots, n\}$ and the natural ordering $1 < 2 < \ldots < n$ is a well order on I. We claim that for any subset $A \subseteq G$ there is a minimal element a.

Indeed, let π_i be a natural projection of G on G_i. Let $A_1 = A$. We have $(A_1)\pi_1 \subseteq G_1$ and since G_1 is well ordered there is a minimal element g_1 of $(A_1)\pi_1$. Next, given A_k and g_k we define $A_{k+1} = \{g \in A_k \mid (k)g = g_k\} \subseteq A_k$. We define g_{k+1} to be a minimal element of $(A_{k+1})\pi_{k+1} \subseteq G_{k+1}$ which exists since G_{k+1} is well ordered. Inductively we obtain the set $\{g_1, \ldots, g_n\}$, where $g_i \in G_i,\ i \in I$. Let $a \in A$ be such that $(a)\pi_i = g_i$ then the construction forces a to be the minimal element of A. Since A was chosen to be an arbitrary non-empty subset of G we have proved that $<_G$ is a well order on $G = \mathrm{Cart}_{i \in I} G_i$, where I is finite.

Now we will prove that $<_G$ is not a well order on G if I is infinite. We do this by constructing an infinite strictly descending sequence.

Choose an infinite countable subset C of I, so $C = \{c_n \mid n \in \mathbb{N}\}$ where $n <_{\mathbb{N}} m$ implies $c_n <_I c_m$. Now for each $n \in \mathbb{N}$ choose $a_n \in G$ such that $(c_n)a_n \neq min_{<_{c_n}} G_{c_n}$ but $(i)a_n = min_{<_i} G_i \ \ \forall i <_I c_n$.

Now $(a_n)_{n=1}^{\infty}$ is an infinite strictly descending sequence in our ordering of G. This proves that $<_G$ is not a well order of G if I is infinite.

The proof of the statement is now completed.

We can now use this proposition to modify theorem 2.1.

Theorem 2.1.1. *Let $(G_i)_{i \in I}$ be a family of well freshly orderable groups and sup-*

pose that $G = Cart_{i \in I} G_i$. Then the following ordering $<_G$ is well fresh on G if I is finite and is fresh but not well fresh if I is infinite. For $g, h \in G$ we write $g <_G h$ if $\exists j \in I$ such that $(j)g <_j (j)h$ and $(i)g = (i)h$ $\forall i <_I j$, where $<_j$ is a well order on G_i and $<_I$ is a well order on I.

Theorem 2.1.1 is an immediate consequence of theorem 2.1 and proposition 2.1.

Next we prove that the properties of being fresh and well fresh are inherited by extensions. Once again we will prove this result in two different ways.

Theorem 2.2. *Let G be a group and let N be a freshly orderable normal subgroup of G. Let $\pi : G \longrightarrow G/N$ be the natural epimorphism, and suppose that G/N is freshly orderable. It follows that G is freshly orderable.*

Proof 1. To simplify notation we suppose that $\pi : g \mapsto \bar{g}$ $\forall g \in G$. Let $<_N$ and $<_{G/N}$ be fresh orderings on N and G/N respectively. Let F be a transversal for the fibres of π. If $g \in G$ then there exists a unique $f_g \in F$ and $n_g \in N$ such that $(g)\pi = (f_g)\pi$ and $g = f_g n_g$.

We define a linear ordering of the underlying set of G as follows. If $g, h \in G$ we write $g < h$ if and only if either $(g)\pi <_{G/N} (h)\pi$ or both $(g)\pi = (h)\pi$ and $n_g <_N n_h$.

We contend that the ordering we have defined is a fresh ordering of G. If $g < h$ by virtue of $(g)\pi <_{G/N} (h)\pi$ we write $g <_\pi h$. Similarly if $(g)\pi = (h)\pi$ we write $g =_\pi h$. Suppose P is a non-empty subset of G and that (for contradiction) there exists $g \in G$ such that $Pg \subseteq \bigcup_{x<g} Px$.

Now

$$Pg \subseteq (\bigcup_{x <_\pi g} Px) \cup (\bigcup_{y =_\pi g,\ y<g} Py).$$

Thus

$$Pg - \bigcup_{x <_\pi g} Px \subseteq \bigcup_{y =_\pi g,\ y < g} Py. \qquad (\star)$$

We have $Pg - \bigcup_{x<_\pi g} Px \neq \emptyset$, because otherwise we would have $Pg \subseteq \bigcup_{x<_\pi g} Px$ and therefore $(Pg)\pi \subseteq (\bigcup_{x<_\pi g} Px)\pi \subseteq \bigcup_{x<_\pi g}(Px)\pi$. This means that $\bar{P}\bar{g} \subseteq \bigcup_{\bar{x}<_{G/N}\bar{g}} \bar{P}\bar{x}$ which contradicts $<_{G/N}$ being fresh on G/N. Let

$$T = (Pf_g - \bigcup_{x<_\pi g} Px) = (Pg - \bigcup_{x<_\pi g} Px)n_g^{-1}$$

so T is not empty.

Here we used the fact that $\bigcup_{x<_\pi g} Px$ is invariant under right multiplication by elements of N. Indeed, $(x <_\pi g) \Leftrightarrow (xN <_{G/N} gN) \Leftrightarrow ((xn)N <_{G/N} gN) \Leftrightarrow (xn <_\pi g)$. Therefore $\forall n \in N$ we have

$$\bigcup_{x<_\pi g} Px = \bigcup_{xn<_\pi g} P(xn) = \bigcup_{x<_\pi g} P(xn) = (\bigcup_{x<_\pi g} Px)n.$$

Now

$$Tn_g = Pg - \bigcup_{x<_\pi g} Px = (Pg - \bigcup_{x<_\pi g} Px) - \bigcup_{x<_\pi g} Px \subseteq$$
$$\bigcup_{y=_\pi g,\ y<g} Py - \bigcup_{x<_\pi g} Px \subseteq \bigcup_{y=_\pi g,\ y<g} (Py - \bigcup_{x<_\pi g} Px)$$

Here we used $(\star)$ for the first inclusion. Next, since $\bigcup_{x<_\pi g} Px$ is invariant under right multiplication by elements of N we have

$$Tn_g \subseteq \bigcup_{n\in N, n<_N n_g} (Pf_g n - \bigcup_{x<_\pi g} Pxn).$$

Thus

$$Tn_g \subseteq \bigcup_{n\in N, n<_N n_g} Tn. \qquad (\star\star)$$

This is almost the necessary contradiction, but we must first translate the set T until it meets N. Choose $w \in G$ such that $M = wT \cap N \neq \emptyset$. Such w obviously exists since we can take $w = nt^{-1}$ where $t \in T,\ n \in N$. We will obtain $n \in (nT \cap N)$, so $nT \cap N \neq \emptyset$. Now the set M has the property we seek, for

$$Mn_g = wTn_g \cap N \subseteq (\bigcup_{n \in N, n <_N n_g} wTn) \cap N \subseteq \bigcup_{n \in N, n <_N n_g} (wTn \cap N) \subseteq$$

$$\bigcup_{n \in N, n <_N n_g} (wTn \cap Nn) \subseteq \bigcup_{n \in N, n <_N n_g} (wT \cap N)n \subseteq \bigcup_{n \in N, n <_N n_g} Mn.$$

Here we used $(\star\star)$ for the first inclusion. The result is absurd since $<_N$ is a fresh order on N and so we are done.

We will now give an alternative proof to the theorem.

In this proof we essentially use the same technique as in the proof 2 of theorem 2.1.

Proof 2. First we invoke the same ordering $<$ on G as in proof 1. We will prove that $<$ is fresh on G.

Let us assume for contradiction that it is not so. Let $P \subseteq G$ be a finite non-empty set of minimal size such that there exists $g \in G$ with $Pg \subseteq \bigcup_{h<g} Ph$. Now we may apply lemma 2.1.1 to obtain a system of equations

$$p_2 g = p_1 h_1$$

$$\dots$$

$$p_{i+1} g = p_i h_i \qquad (\star)$$

$$\dots$$

$$p_1 g = p_{n+1} g = p_n h_n$$

where $h_i < g$ $(i \in \{1,\dots,n\})$, $p_i \neq p_j$ if $i \neq j$ $(i,j \in \{1,\dots,n\}), p_{n+1} = p_1$. We now focus on $h_i < g$ $(i \in \{1,\dots,n\})$. Recalling the definition of the order $<$ and of the map π we conclude that there are only the following two possibilities here:

(1). $\forall h_i$ $(i \in \{1,\dots,n\})$ $(h_i)\pi = (g)\pi$.

(2). $\forall h_i$ $(i \in \{1,\dots,n\})$ $(h_i)\pi \leq_{G/N} (g)\pi$ and $\exists j \in \{1,\dots,n\}$ such that $(h_j)\pi <_{G/N} (g)\pi$.

If (1) is the case then we have $\forall h_i$ $(i \in \{1,\dots,n\})$ $h_i = f_g n_{h_i}$. We can deduce from the system $(\star)$ that in this case we have $\forall p_i$ $(i \in \{1,\dots,n\})$ $p_i = f_p n_{p_i}$. Now we can rewrite $(\star)$ as follows.

$$f_p n_{p_2} f_g n_g = f_p n_{p_1} f_g n_{h_1}$$

$$\dots$$

$$f_p n_{p_{i+1}} f_g n_g = f_p n_{p_i} f_g n_{h_i}$$

$$\dots$$

$$f_p n_{p_1} f_g n_g = f_p n_{p_{n+1}} f_g n_g = f_p n_{p_n} f_g n_{h_n}$$

where $\forall i \in \{1,\dots,n\}$ $n_{h_i} <_N n_g$. Multiplying by $f_g^{-1} f_p^{-1}$ on the left we obtain

$$(f_g^{-1} n_{p_2} f_g) n_g = (f_g^{-1} n_{p_1} f_g) n_{h_1}$$

$$\dots$$

$$(f_g^{-1} n_{p_{i+1}} f_g) n_g = (f_g^{-1} n_{p_i} f_g) n_{h_i}$$

$$\dots$$

$$(f_g^{-1} n_{p_1} f_g) n_g = (f_g^{-1} n_{p_{n+1}} f_g) n_g = (f_g^{-1} n_{p_n} f_g) n_{h_n}$$

where $\forall i \in \{1, \dots, n\}$ $n_{h_i} <_N n_g$. Since N is normal in G we have $(f_g^{-1} n_{p_i} f_g) = \widehat{n}_{p_i} \in N$. Thus we finally have

$$\widehat{n}_{p_2} n_g = \widehat{n}_{p_1} n_{h_1}$$

$$\dots$$

$$\widehat{n}_{p_{i+1}} n_g = \widehat{n}_{p_i} n_{h_i}$$

$$\dots$$

$$\widehat{n}_{p_1} n_g = \widehat{n}_{p_{n+1}} n_g = \widehat{n}_{p_n} n_{h_n}$$

where $\forall i \in \{1, \dots, n\}$ $n_{h_i} <_N n_g$.

Thus we obtained $(\{\widehat{n}_{p_1}, \dots, \widehat{n}_{p_n}\},\ n_g)$ - a counter-example to the ordering $<_N$ being fresh on N. Therefore case (1) is in fact impossible.

We now assume that case (2) occurs. We apply the map π to the system $(\star)$ to obtain

$$\overline{p_2}\bar{g} = \overline{p_1}\overline{h_1}$$

$$\dots$$

$$\overline{p_{i+1}}\bar{g} = \overline{p_i}\overline{h_i} \tag{$\star\star$}$$

$$\dots$$

$$\overline{p_1}\bar{g} = \overline{p_{n+1}}\bar{g} = \overline{p_n}\overline{h_n}$$

where $\overline{h_i} \leq_{G/N} \bar{g}$ $(i \in \{1, \dots, n\})$ and $\exists j \in \{1, \dots, n\}$ such that $\overline{h_j} <_{G/N} \bar{g}$. Also, $\overline{p_1}, \dots, \overline{p_n}$ are not necessarily all different (since π is not injective unless $N = 1$, but in this case the theorem is trivially true.)

We are now in the position to apply lemma 2.1.2 using G/N as the group G $(P)\pi$ as the set P, $<_{G/N}$ as the order $<$ and $(\star\star)$ as the system $(\bigwedge)$ of equations in the lemma respectively.

We conclude that $<_{G/N}$ is not fresh on G/N which is a contradiction. Therefore case (2) also leads to a contradiction. Thus our assumption was wrong and so $<$ is fresh on G.

The theorem is now proved.

In order to show that formation of extensions preserves not only the property of groups being fresh but also being well fresh we will need the following proposition.

Proposition 2.2. *If $<_N$ is a well order of a normal subgroup N of a group G and $<_{G/N}$ is a well order of the quotient group G/N then the following order $<$ of G is a well order.*

For every $h,\ g \in G$ we say that $h < g$ if

$(h)\pi <_{G/N} (g)\pi$ or

$(h)\pi = (g)\pi$ and $n_h <_N n_g$

where π is the natural map $G \longrightarrow G/N,\ g \mapsto gN$ and every $g \in G$ has a unique representation as $g = f_g n_g$ where $n_g \in N,\ f_g \in F$ - a transversal of the fibres of π.

Proof. Clearly $<$ is a linear order. We will prove that it is in fact a well order.

We consider an arbitrary non-empty set A of G. Our ambition is to prove that $\exists a \in A$ such that $a < g \ \ \forall g \in A - \{a\}$.

We consider $(A)\pi$ which is a non-empty subset of G/N which is well ordered. Therefore there is a minimal element $\hat{f}N$ of $(A)\pi$. Next we consider such elements $g \in A$ that $f_g = \hat{f}$ or in other words $\{g \in A \mid (g)\pi = (\hat{f})\pi\}$. They all are of the form $\hat{f}n_g$.

Now let $M = \{n \in N \mid \exists g \in A \text{ such that } g = \hat{f}n\} \subseteq N$. Now $M \neq \emptyset$ because $n_g \in M$. Since N is well ordered there is a minimal element $\hat{n}$ of M. We claim that $a = \hat{f}\hat{n}$.

Indeed, $\forall g \in A$ we have $g = f_g n_g$. Since $(a)\pi = \hat{f}N$ which is a minimal element of $(A)\pi$ then $(a)\pi = \hat{f}N \leq_{G/N} f_g N = (g)\pi$.

If $(a)\pi <_{G/N} (g)\pi$ then $a < g$.

If $(a)\pi = (g)\pi = \hat{f}N$ then $\hat{n} \leq_N n_g$ since in this case $f_g = \hat{f}$ and so $n_g \in M$ of which $\hat{n}$ is a minimal element.

If $\hat{n} <_N n_g$ then we have $a < g$. If $\hat{n} = n_g$ then $a = g$.

Thus we proved that a is a minimal element of A. Therefore $<$ is indeed a well order.

We can now use this proposition to modify theorem 2.2.

Theorem 2.2.1. *Let G be a group, N be its normal subgroup and G/N be the quotient group. If N and G/N are well freshly orderable groups then G is also well freshly orderable.*

Theorem 2.2.1 immediately follows from theorem 2.2 and proposition 2.2.

From both theorem 2.1 and 2.2 we can derive the following result.

Theorem 2.3. *Any free group is freshly orderable.*

Proof. We know that the infinite cyclic group is freshly orderable as an immediate consequence of theorem 1.2 of chapter 1. Also by theorem 2.2 any poly-C_∞ group is freshly orderable. It now follows by theorem 2.1 and the fact that a subgroup of a fresh group is fresh (see p. 59) that any subgroup of a Cartesian product of poly-C_∞ groups is freshly orderable.

Thus it suffices to show that any free group F is residually poly-C_∞ since that means there is a collection N_λ $(\lambda \in \Lambda)$ of normal subgroups of F such that F/N_λ is poly-C_∞ $(\forall \lambda \in \Lambda)$ and $\bigcap_{\lambda\in\Lambda} N_\lambda = 1$. Now the obvious map $F \longrightarrow \mathrm{Cart}_{\lambda\in\Lambda} F/N_\lambda$ is a monomorphism and so we are done. It remains to show that a free group F is residually poly-C_∞ and we do this as follows.

If F is finitely generated this follows from a couple of results in Philip Hall's Edmonton Notes [8]. On page 40 theorem 5.8 tells us that $\bigcap_{n=1}^{\infty} \gamma_n(F) = 1$ where F is any free group and $\gamma_n(F)$ is the nth term of the lower central series of F.

If F is finitely generated then $F/\gamma_n(F)$ is torsion-free since it is poly-free abelian by the corollary to theorem 5.6 on page 38 of [8].

Now it suffices to show that the free group F is residually a finitely generated free group. Let F be free on X and suppose $g \in F - \{1\}$. Let $Y \subseteq X$ be the finite subset of X consisting of the letters involved in the irreducible form of g.

Now $\langle Y \rangle$ is free on Y since irreducible words on Y are irreducible words on X. The map $X \longrightarrow \langle Y \rangle$ where $y \mapsto y$ if $y \in Y$ and $x \mapsto 1$ if $x \notin Y$ defines an epimorphism $\theta_Y : F \longrightarrow \langle Y \rangle$ onto the finitely generated free group $\langle Y \rangle$ and $(g)\theta_Y = g \neq 1$. Thus we are done.

Next, we shall give a direct proof of this theorem. While being a useful existence argument, the ordering on the free group implicit in this theorem is less than transparent.

We will demonstrate a very natural ordering on the free group which is fresh.

Moreover it is well fresh. This allows us to strengthen theorem 2.3 by stating that any free group is in fact well fresh.

Let $X^{-1} = \{x^{-1} | \; x \in X\} \subseteq G$ and put $\Sigma = X \cup X^{-1}$. We now use the principle of well ordering to place a well order on Σ and then extend this to a well order of $\Sigma^\star$ via the short-lex ordering. We note that here $\Sigma^\star$ by definition consists of words in symbols from X and X^{-1} which are formal objects, not elements of G.

We remind that the short-lex ordering is the following one:

$\forall \omega_1, \; \omega_2 \in \Sigma^\star$ such that $\omega_1 = x_1 \ldots x_{k_1}, \; \omega_2 = y_1 \ldots y_{k_2}$

$$\omega_1 \sqsubset \omega_2 \Leftrightarrow$$

either $k_1 < k_2$ or $k = k_1 = k_2$ and $\exists j$ from $\{1, \ldots, k\}$ such that $x_j \sqsubset y_j, \; x_i = y_i \;\; (\forall i < j)$.

As is well known, the natural map $\pi : \Sigma^\star \to G$ is a monoid homomorphism, and if $g \in G$ then $(g)\pi^{-1}$ contains a unique irreducible word.

We remind the reader that $\omega \in \Sigma^\star$ is irreducible if no element of X is juxtaposed to its inverse in the word.

Let the set of irreducible words of $\Sigma^\star$ be denoted $Irr(\Sigma^\star)$, so π induces a bijection $\hat{\pi} : Irr(\Sigma^\star) \to G$.

By the length $l(g)$ of $g \in G$ we will denote the number of symbols of Σ in $\hat{\pi}^{-1}(g)$.

We use this bijection to construct a well ordering of the underlying set G as follows:

If $g_1, \; g_2 \in G$ we write $g_1 < g_2$ if and only if $(g_1)\hat{\pi}^{-1} < (g_2)\hat{\pi}^{-1}$ where the final use of the symbol $<$ is denoting the short-lex ordering.

In lemma 2.4.3 we will give a formal proof of the fact that short-lex ordering on the free group is indeed a well ordering.

Suppose $g \in G$, so $(g)\pi^{-1} = y_1 y_2 ... y_n \in Irr(\Sigma^*)$, where each $y_i \in X \cup X^{-1}$. The reduced form of g is the expression $(y_1)\hat{\pi}(y_2)\hat{\pi}...(y_n)\hat{\pi} \in G$. If we identify $y \in \Sigma$ with $y \in G$ via $\hat{\pi}$ we can write the reduced form of g as $y_1 y_2 ... y_n$ where $y_i y_{i+1} \neq 1_G \quad \forall i < n$. For example, the reduced form of $x^{-1}xyx$ is yx.

We define the (right) Cayley graph of G as follows:

The graph $\Gamma = (V, E)$ has $V = G$; the pair $(g_1, g_2) \in E$ if and only if there exists $x \in X$ such that $g_1 x = g_2$. This gives us a directed graph and we may label the directed edges with corresponding generators.

For example, if $X = \{x, y\}$ where $x \neq y$ and $G =< X|- >$ then a fragment of the Cayley graph is as shown in diagram 1 below.

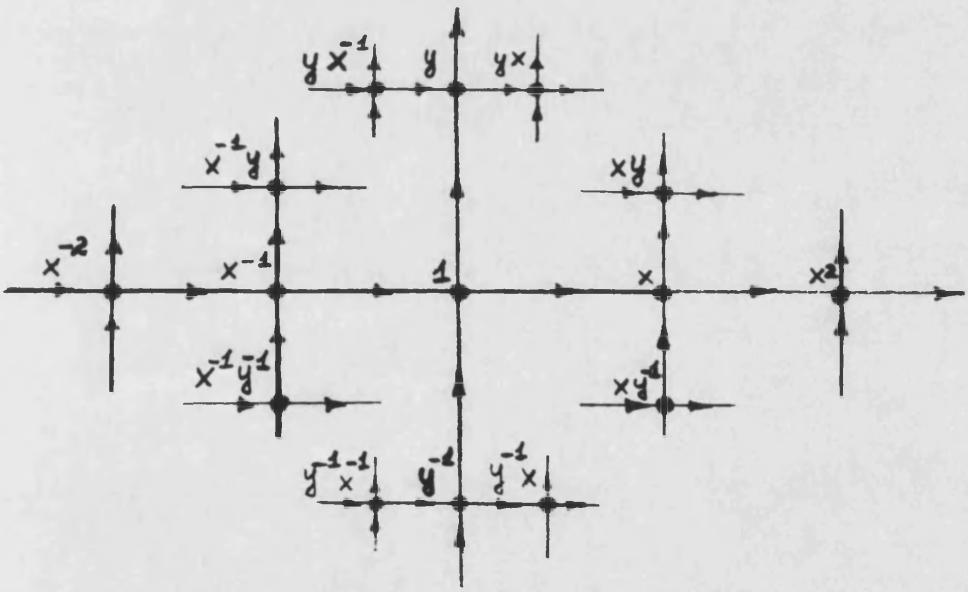


Diagram 1

Going back to the general case, it is clear that the Cayley graph of our group G is a tree, which means that it is connected and has no non-trivial circuits. This amounts to the assertion that a non-empty irreducible word does not represent the identity element of G.

Geometrically, an irreducible word of $\Sigma^\star$ - or the reduced form of some $g \in G$ is simply a geodesic path in the Cayley graph from 1 to g.

We will now state the result.

Theorem 2.4. *A short-lex ordering $<$ on $G =< X| - >$ is a fresh well ordering.*

Proof. Suppose for contradiction that $<$ is not fresh on G. This means that there exists a counter-example to the theorem. In other words there is a pair (P, g) such that

$$Pg \subseteq \bigcup_{h \in G,\ h<g} Ph$$

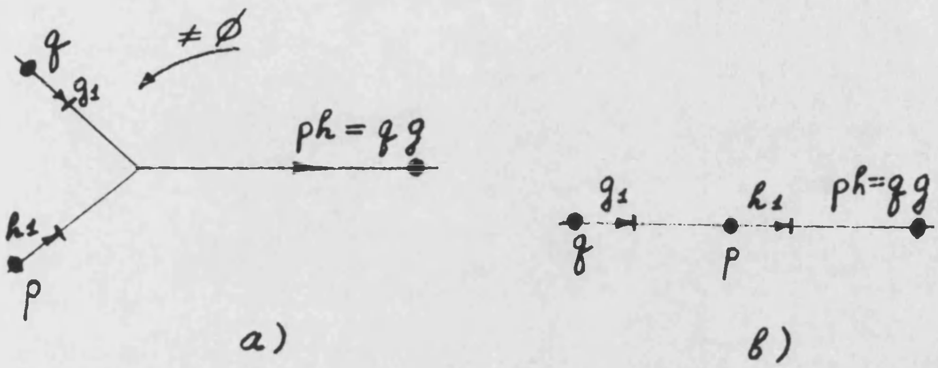
where $P \subseteq G,\ P \neq \emptyset$ and P is finite.

We note that $g \neq 1$ otherwise we would have $\emptyset \neq Pg = P \subseteq \bigcup_{h<1} Ph = \emptyset$.

Among all counter-examples we will choose one with $|P|$ of minimal size. We will now apply lemma 2.1.1 (a) using our short-lex ordering $<$ as the linear ordering $<$ in the lemma. Clearly our chosen P satisfies the conditions of the lemma. Thus we can draw the following corollaries using p and q as in lemma 2.1.1 (a).

Lemma 2.4.1. *In the Cayley graph Γ of G the geodesic path from p to q passes through no other points of P as vertices.*

Proof. We have the following pictures given in diagrams 2a and 2b. The diagram 2a expresses the general case. Obviously the top left branch there can never be empty. However, it may be that the bottom left branch is empty, in which case the correct diagram is 2b.



Diagrams 2a and 2b

If $p' \in P$ is strictly between p and q then let $h' = red({p'}^{-1}qg)$ so $p'h' = qg$ and the length of h' is the length of the geodesic path in the Cayley graph from p' to qg. Since $l(h') < \max\ (l(g),\ l(h)) = l(g)$ this would violate lemma 2.1.1.

Lemma 2.4.2. *The geodesic path from p to q has the last edge labelled g_1^{-1} where $g = g_1 g_2 ... g_t$ is the reduced form of g.*

(Here an edge taken in reverse order is deemed to have the inverse label, so the form of $p^{-1}q$ ends with g_1^{-1}.) We can now continue the proof of the theorem.

Among all counter-examples to the theorem of the form (P', g') consider those where $|P'| = n$ is minimal. Now select from these a counter-example (P, g) with g minimal in the short-lex ordering.

We will now apply lemma 2.1.1 (b) using our short-lex ordering $<$ as the linear ordering $<$ in the lemma. Clearly our P and g also satisfy the conditions of the lemma. Thus we have the equations $p_{i+1}g = p_i h_i$ with $h_i < g$ $(i = 1, \ldots, n)$, $p_i \neq p_j$ if $i \neq j$ $(i, j \in 1, \ldots, n)$, $p_{n+1} = p_1$.

Join p_i to p_{i+1} via a geodesic path corresponding to the reduced form of $p_i^{-1} p_{i+1}$.

This path must "end in g_1^{-1}" by lemma 2.4.1. The path from p_{i+1} to p_{i+2} must start by backtracking along the path from p_i to p_{i+1} as the following diagram 3 shows.

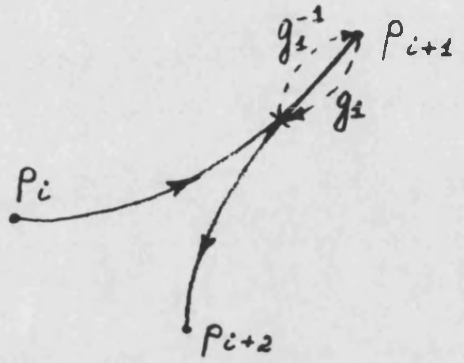


Diagram 3

This is because assuming the contrary leads us to a contradiction. Indeed, let us suppose that the first edge on the geodesic path from p_{i+1} to p_{i+2} is not labelled with g_1 but with some $x \neq g_1$. Moving through the points of P by going from one to the next one along the corresponding geodesic we will not encounter p_{i+1} anywhere *en route* according to lemma 2.4.1. But since the points of P in their arranged order form a cycle we will eventually reach p_{i+1} again entering via the geodesic from p_i. Thus the last ingoing edge is labelled g_1^{-1} but the first outgoing edge is labelled $x \neq g_1$ and there is no p_{i+1} anywhere else in the loop. So, we created a non-trivial cycle on the graph Γ which is a tree.

This is pictured in diagram 4 below.

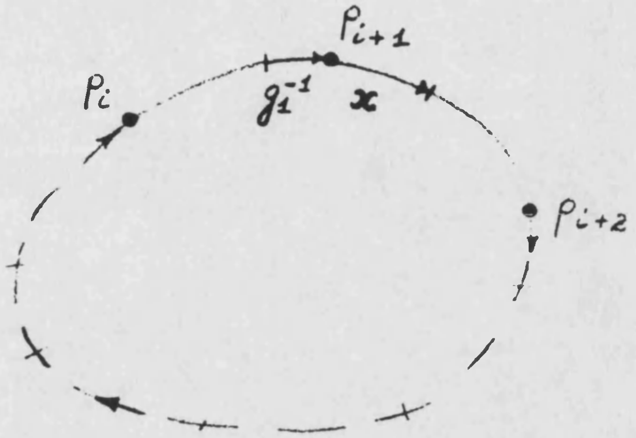


Diagram 4

This is the required contradiction.

Thus every reduced form of $p_i^{-1}p_{i+1}$ must begin with g_1. So we now know that a geodesic path from p_i to p_{i+1} starts with g_1 and ends with g_1^{-1}. Therefore it has length at least 3 (see diagram 5 below).

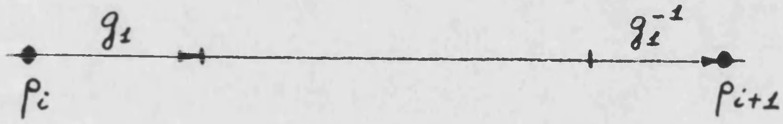


Diagram 5

Now we are ready to complete the proof by constructing a counter-example to the theorem "smaller" than (P, g) in terms of g.

Let $g = g_1\hat{g}$, where $g_1 \neq 1$ since $g \neq 1$ as we mentioned earlier. Let $\hat{P} = Pg_1 = \{p_1g_1, ..., p_ng_1\}$. We will show that $(\hat{P}, \hat{g})$ is a "smaller" counter-example than (P, g) with respect to g.

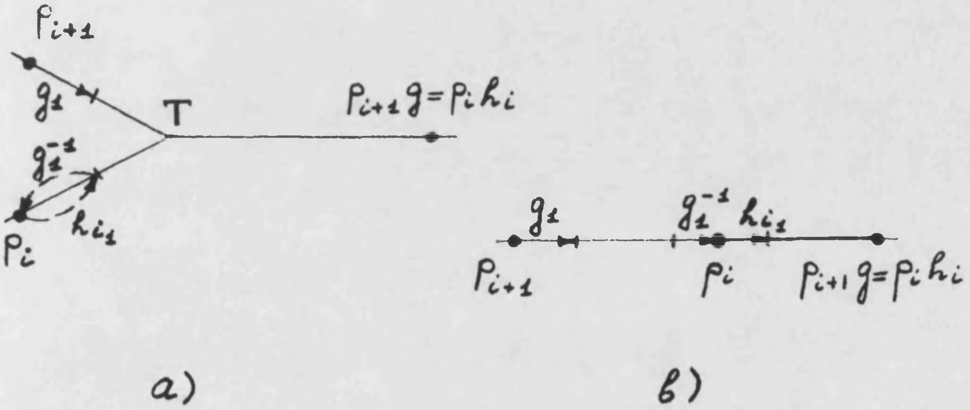
We know that $p_{i+1}g = p_ih_i \quad \forall i \leq n$. We will write $h_i = h_{i_1}\hat{h}_i$, where it is a reduced form of h_i. (Here h_{i_1} is generally a single letter but h_{i_1} can be equal to 1 if and only if $h_i = 1$).

If $h_{i_1} = g_1$ then clearly $\hat{h_i} < \hat{g}$ and we have

$$(p_{i+1}g_1)\hat{g} = (p_i h_{i_1})\hat{h_i} = ((p_i g_1)\hat{h_i}, \ \hat{h_i} < \hat{g}.$$

Now we consider the case $h_{i_1} \neq g_1$. In the general case (as shown in diagram 6a below) as we know, the branch (p_{i+1}, T) is not empty. That is to say p_{i+1} is not on the geodesic $(p_i, p_i h_i)$. However, (p_i, T) may or may not be empty.

If (p_i, T) is not empty, that is to say if p_i is not on the geodesic $(p_{i+1}, \ p_{i+1}g)$ then $h_{i_1} = g_1$. Thus if $h_{i_1} \neq g_1$ then p_i is on the geodesic $(p_{i+1}, \ p_{i+1}g)$ or in other words we have the situation as shown in diagram 6b.



Diagrams 6a and 6b

So, if $h_{i_1} \neq g_1$, we have that p_i is on the geodesic $(p_{i+1}, \ p_{i+1}g)$ (see (6b)) and we can write

$$(p_{i+1}g_1)\hat{g} = (p_i g_1)(g_1^{-1}h_i).$$

In this case we have

$$l(\hat{g}) = l(g) - 1 = l(p_{i+1}, p_i) + l(h_i) - 1 \geq 3 + l(h_i) - 1 = l(h_i) + 2 > l(h_i) + 1 = l(g_1^{-1}h_i).$$

So, we have $(p_{i+1}g_1)\hat{g} = (p_i g_1)(g_1^{-1}h_i)$ with $g_1^{-1}h_i < \hat{g}$.

Thus in any event we have

$$(p_{i+1}g_1)\hat{g} = (p_i g_1)\hat{h_i}, \ \hat{h_i} < \hat{g}, \ \forall i \leq n.$$

In other words we have

$$\hat{P}\hat{g} \subseteq \bigcup_{\hat{h} \in G, \ \hat{h} < \hat{g}} \hat{P}\hat{h}.$$

Hence $(\hat{P}, \hat{g})$ is indeed a counter-example where $|\hat{P}| = |P|$ but $\hat{g} < g$ which violates the minimality of the choice of g. Thus we have proved that $<$ is a fresh ordering on G. To complete the proof of the theorem we will now show that $<$ is also a well ordering of G. Of course, this result is well known and admits a variety of proofs.

Lemma 2.4.3. *The short-lex ordering $<$ is a well ordering of the free group F.*

Proof. Let A be a non-empty subset of the free group F (with words written in irreducible form.) Let B be the set of elements of A of minimal word length n, so $B \neq \emptyset$.

Let $B_0 = B$ and define $B_i \ \ (i \geq 1)$ inductively. B_i consists of those elements of B_{i-1} which have minimal ith element. Now $B_n = \{a\}$ must be a singleton set and a is the minimal element of A.

Thus the short-lex ordering is indeed a well ordering of the free group F.

The theorem is now proved.

Corollary 2.4. *If G is a free group, $<$ is a short-lex ordering, $g \in G$ and $h_i < g$ $(i = 1, ..., n-1)$ then*

$$gh_{n-1}^{-1}gh_{n-2}^{-1}...h_1^{-1}g > g.$$

Proof. If

$$gh_{n-1}^{-1}gh_{n-2}^{-1}...h_1^{-1}g = h_n < g$$

then we let $p_n = 1$ and define p_i for $i < n$ via $p_{i+1}g = p_ih_i$, in other words $p_i = p_{i+1}gh_i^{-1}$.

Thus we will obtain $p_1 = gh_{n-1}^{-1}g...gh_1^{-1}$, so

$$p_1g = gh_{n-1}^{-1}g...h_1^{-1}g = h_n = p_nh_n.$$

So we have $(P = \{p_1, ..., p_n\}, g)$ a counter-example to our theorem since $h_i < g \ (1 \leq i \leq n)$.

If on the other hand we assume that

$$gh_{n-1}^{-1}g...h_1^{-1}g = g$$

we will get $gh_{n-1}^{-1}g...h_2^{-1}g = h_1 < g$, but we have just shown that a construction like this can not be strictly less than g.

Therefore we have proved that $gh_{n-1}^{-1}g...h_1^{-1}g > g$.

As we demonstrated earlier the property of being fresh is inherited by Cartesian products and extensions. Clearly this property is also inherited by subgroups as well as is the property of being well fresh.

Here is another group construction which preserves the property of being well fresh. Our previous results together with the Kurosh subgroup theorem make the proof of this one very short.

Theorem 2.5. *The free product $A * B$ of well freshly orderable groups A and B is well freshly orderable.*

Proof. We consider a natural epimorphism $\phi : A * B \longrightarrow A \times B$.

By the Kurosh subgroup theorem (explained below) the Kernel of ϕ is a free group. Thus by the first isomorphism theorem we have

$$A * B/Ker\phi \cong A \times B$$

Since A and B are well fresh then by theorem 2.1.1 $A \times B$ is also well fresh. Now we apply theorem 2.2.1 to obtain that $A * B$ is also well fresh.

The theorem is now proved.

We remind the reader that the Kurosh subgroup theorem (see Kurosh [10]) states that every subgroup H of a free product G, of groups A and B is itself a free product of a free group F and the intersection of H with certain conjugates of A and B. An accessible proof can be found in Magnus, Karrass and Solitar [11] p. 243.

Next we will draw yet another result as a corollary to theorems 2.1 and 2.2. This time it concerns the property of groups being locally well fresh.

Theorem 2.6. *Torsion-free abelian groups are locally well freshly orderable.*

As we will see later in theorem 3.1 and corollary 3.1.2 of chapter 3 torsion-free abelian groups are freshly orderable. But whether a fresh ordering on them is a well ordering depends, generally speaking, on every particular group and its structure (see the proof of theorem 3.1). Here we will make sure that the ordering we construct for every given P is well fresh for P. We remind the reader that the definition of a linear ordering of a group G being well fresh for a set P on G is given in the introduction as part of definition B (p. 10).

Proof. Let P be a non-empty finite subset of a torsion-free abelian group A. We will construct an ordering $<$ of A which will be well fresh for P. This means that

$<$ will be a well ordering with the property that

$$\forall g \in A \quad Pg \nsubseteq \bigcup_{h<g} Ph.$$

First we consider a subgroup $< P >$ generated by the set P. Since P is finite then $< P >$ is finitely generated. Also $< P >$ obviously inherits the properties of A being abelian and torsion-free. Thus $< P >$ is a finitely generated torsion-free abelian group. From the theory we can now deduce that $< P >$ is in fact a finitely generated free abelian group.

Since $< P >$ is free abelian we can use theorem 1.3 of chapter 1 to conclude that $< P >$ is well fresh. We will denote a well fresh order on $< P >$ by $<_{<P>}$.

Now we consider a set of right cosets of $< P >$ in A. We will denote it by $< P > \backslash A$. Let T be a transversal of $< P > \backslash A$. We will impose a well order on T, denoting it by $<_T$.

Next let ψ be the map: $A \longrightarrow T$ that sends each element of A into the representative in T of its right coset of $< P >$.

We will now describe an ordering $<$ of A: $h < g$ exactly when the following conditions hold.

Either $(h)\psi <_T (g)\psi$

or $(h)\psi = (g)\psi$ and $h((h)\psi)^{-1} <_{<P>} g((g)\psi)^{-1}$.

We will prove that the order $<$ is well fresh for P. First we show that $<$ is P-fresh and then we show that it is also a well order of A.

Clearly $<$ is a linear order of A. We assume for contradiction that it is not P-fresh. This means that $\exists g \in A$ such that

$$Pg \subseteq \bigcup_{h<g} Ph.$$

In more detail it means that

$$Pg \subseteq (\bigcup_{(h)\psi <_T (g)\psi} Ph) \cup (\bigcup_{(h)\psi=(g)\psi,\ h<g} Ph). \qquad (\star)$$

Therefore

$$Pg - (\bigcup_{(h)\psi <_T (g)\psi} Ph) \subseteq \bigcup_{(h)\psi=(g)\psi,\ h<g} Ph.$$

But $Pg \subseteq < P > g,\ Ph \subseteq < P > h$ and $< P > g \cap < P > h = \emptyset$ if $(h)\psi \neq (g)\psi$. Thus $Pg \cap (\bigcup_{(h)\psi <_T (g)\psi} Ph) = \emptyset$ and so we have $Pg \subseteq \bigcup_{(h)\psi=(g)\psi,\ h<g} Ph$. In other words we have

$$Pg \subseteq \bigcup_{(h)\psi=(g)\psi,\ h((h)\psi)^{-1} <_{<P>} g((g)\psi)^{-1}} Ph. \qquad (\star\star)$$

We will now denote $(h)\psi = (g)\psi$ by t and multiply both parts of $(\star\star)$ by t^{-1} on the right. We obtain

$$Pgt^{-1} \subseteq \bigcup_{(h)\psi=(g)\psi=t,\ ht^{-1} <_{<P>} gt^{-1}} Pht^{-1}.$$

Now we denote gt^{-1} and ht^{-1} by x and y respectively to obtain

$$Px \subseteq \bigcup_{y <_{<P>} x} Py$$

where $P \subseteq < P >$ and $x,\ y \in < P >$ (indeed $x = gt^{-1},\ g \in < P > t$, so $x \in < P >$; in the same way $y \in < P >$).

Thus we came to contradiction with $<_{<P>}$ being fresh on $< P >$. This proves that our assumption was wrong and so $<$ is P-fresh on A.

Finally we will prove that $<$ is also a well order of A. We have to show that for any non-empty subset $M \subseteq A$ there is a minimal element m of M with respect to $<$.

We know that each element a of the group A can be expressed uniquely as $a = pt$, where $p \in < P >,\ t \in T,\ (t = (a)\psi)$.

We consider the set $S = \{t \in T \mid \exists x \in M \text{ such that } t = (x)\psi\}$. Since $S \subseteq T$ and T is well ordered there is a minimal element $s \in S$. Now we will consider the set $L = \{x \in M \mid (x)\psi = s\}$. Since $Ls^{-1} \subseteq < P >$ and since $< P >$ is well ordered Ls^{-1} has a minimal element l. Now ls is clearly the minimal element m of M and so $<$ is a well order.

Thus we can conclude that the class of freshly orderable groups is closed under taking subgroups, Cartesian products, extensions and free products. We have shown that free groups are well fresh and torsion free abelian groups are locally well fresh. It was also demonstrated that fresh groups are torsion-free.

CHAPTER 3

In this chapter we describe a large family of groups which are freshly orderable. Also we prove that the group ring of a fresh group has no zero-divisors.

A group G is called right(left)-orderable if there is a linear ordering $<$ of G such that it is invariant under the group multiplication on the right(left).

A group G is called orderable if there is a linear ordering $<$ of G such that it is invariant under group multiplication on each side.

Theorem 3.1. *If G is a right(left)-orderable group with respect to a linear ordering $<$ of G then G is freshly orderable with respect to $<$.*

Proof. We assume for contradiction that $<$ is not fresh on G. Therefore the set $\Omega = \{P \mid P \subset G,\ P \neq \emptyset,\ |P| < \infty,\ < \text{ not fresh for } P\}$ is not empty. Let $P \in \Omega$ be such that $|P|$ is of a minimal size among elements of Ω. Let $g \in G$ be such that $Pg \subseteq \bigcup_{h<g} Ph$. Next we apply lemma 2.1.1 from chapter 2 to arrange the elements of P into the following system of equations.

$$p_2 g = p_1 h_1$$

$$\dots$$

$$p_{i+1} g = p_i h_i$$

$$\dots$$

$$p_1 g = p_{n+1} g = p_n h_n$$

where $p_i \neq p_j$ if $i \neq j$ for $(i, j \in 1, \dots, n)$, $p_{n+1} = p_1$ and $h_i \in G,\ h_i < g,\ i \in \{1, \dots, n\}$.

From the last equation we express p_1 as $p_n h_n g^{-1}$ and substitute it to the first equation to obtain $p_2 g = (p_n h_n g^{-1})h_1 = p_n(h_n g^{-1} h_1)$. We denote $h_n g^{-1} h_1$ by h.

It is easy to see that $h < g$. Indeed, we know that $h_1,\ h_n < g$. Now, if G is right-orderable we multiply both parts of inequality $h_n < g$ by $g^{-1}h_1$ on the right to obtain $h = h_n g^{-1} h_1 < g g^{-1} h_1 = h_1 < g$. If G is left-orderable we multiply both parts of inequality $h_1 < g$ by $h_n g^{-1}$ on the left to obtain $h = h_n g^{-1} h_1 < h_n g^{-1} g = h_n < g$.

Thus we have the following system of equations

$$p_2 g = p_n h$$

$$\dots$$

$$p_{i+1} g = p_i h_i$$

$$\dots$$

$$p_n g = p_{n-1} h_{n-1}$$

where $p_i \neq p_j$ if $i \neq j$ for $(i, j \in 2, \dots, n)$ and $h_i \in G,\ h_i < g,\ i \in \{3, \dots, n\},\ h < g$.

This means that $\hat{P} g \subseteq \bigcup_{h<g} \hat{P} h$ where $\hat{P} = \{p_2, \dots, p_n\} \subsetneq P = \{p_1, \dots, p_n\}$. But this is a contradiction to the minimality of P.

Thus $<$ is fresh on G and the theorem is now proved.

The proof actually shows that a stronger statement is true.

Corollary 3.1. *If G is a group with the following property: ($\forall g \in G$ and $\forall x_1$, $x_2 \in G$ such that $x_1,\ x_2 < g$ we have $x_1 g^{-1} x_2 < g$ where $<$ is a linear ordering on G) then $<$ is fresh on G.*

If the order $<$ of a right(left) orderable group G is also a well order then, of course, G is not merely fresh, but also well fresh.

We will demonstrate below that the class of orderable groups is genuinely smaller than that of right(left)orderable groups. For this smaller class theorem 3.1 can be proved directly, as follows, using both right and left orderability.

Proposition 3.1. *If a group G is orderable with respect to a linear ordering $<$ then G is fresh with respect to $<$.*

Proof. We assume for a contradiction that $<$ is not fresh on G. This means that $\exists P \subset G,\ P \neq \emptyset,\ |P| < \infty$ and $\exists g \in G$ such that

$$Pg \subseteq \bigcup_{h<g} Ph \qquad (\star).$$

Let $m = \max_{<} P$. It follows from $(\star)$ that $\exists p \in P,\ h \in G,\ h < g$ such that $mg = ph$. Thus $m = phg^{-1}$.

Now, since G is right-orderable we have $(h < g) \Rightarrow (hg^{-1} < gg^{-1} = 1)$. Also, since G is left-orderable we have $(hg^{-1} < 1) \Rightarrow (p(hg^{-1}) < p \cdot 1)$.

Therefore we obtain $m = phg^{-1} < p$. On the other hand $m \geq p\ \ \forall p \in P$ since m is maximal in P. The contradiction proves that $<$ is indeed fresh on G.

Next, we will give two corollaries to theorem 3.1 concerning some particular groups.

Consider the group G of what Пeter Neumann calls automorphisms of the rational world (see [15]). Thus $(\alpha \in G) \Leftrightarrow \forall x,\ y \in \mathbb{Q}\ \ (x < y) \Rightarrow ((x)\alpha < (y)\alpha)$, where $<$ is the natural ordering of $\mathbb{Q}$. Put a well ordering $<_{\mathbb{Q}}$ on $\mathbb{Q}$. We impose an ordering $\sqsubset$ on G as follows.

$\forall \alpha, \beta \in G \quad \alpha \sqsubset \beta$ if and only if

$\exists q_t \in \mathbb{Q}$ such that $(q_t)\alpha < (q_t)\beta$ and

$$(q_i)\alpha = (q_i)\beta \ \forall q_i <_{\mathbb{Q}} q_t, \ q_i \in \mathbb{Q}.$$

Corollary 3.1.1. *The ordering $\sqsubset$ of the group G is fresh.*

Proof. The ordering $\sqsubset$ of G is clearly linear. We claim that G is right orderable with respect to $\sqsubset$.

Indeed, if $\alpha \sqsubset \beta$ in G then $\forall g \in G \ \ \alpha g \sqsubset \beta g$, since $\exists q_t \in \mathbb{Q}$ such that $(q_t)\alpha < (q_t)\beta$ and that $(q_i)\alpha = (q_i)\beta \ \forall q_i <_{\mathbb{Q}} q_t, \ q_i \in \mathbb{Q}$.

So $(q_t)(\alpha g) < (q_t)(\beta g)$ and $(q_i)(\alpha g) = (q_i)(\beta g) \ \forall q_i <_{\mathbb{Q}} q_t, \ q_i \in \mathbb{Q}$.

Now we can use theorem 3.1 to conclude that $\sqsubset$ is fresh on G.

Next, using the right orderable group G we can illustrate that the collection of right(left) orderable groups is strictly larger than that of orderable groups.

We observe that there are $x, \ y \in G$ such that $x \sqsubset y$ and $x^2 = y^2$. On the other hand in an orderable group we always have $(x < y) \Rightarrow (x^2 < y^2)$ for any elements $x, \ y$ of the group. Indeed, $(x < y) \Rightarrow (xx < yx)$ by the right orderability and $(x < y) \Rightarrow (yx < yy)$ by the left orderability. Therefore $(x < y) \Rightarrow (x^2 < yx < y^2)$ for any $x, \ y$ from an orderable group.

Thus existence of $x, \ y \in G$ such that $x \sqsubset y$ and $x^2 = y^2$ proves that the right orderable group G is not orderable with respect to the ordering $\sqsubset$. We build these functions $x, \ y \in G$ as follows.

Let $h : [0, 1/3) \longrightarrow [1/3, 1)$ be the bijection defined by $x \mapsto 2x + 1/3 \forall \, x \in [0, 1/3)$. Define a map $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ using h and the integer-part function as follows.

$$f : q \mapsto [q] + (q - [q])h$$

if $q - [q] \in [0, 1/3)$ and

$$q \mapsto [q] + (q - [q])h^{-1} + 1$$

if $q - [q] \in [1/3, 1)$.

Notice $f \circ f : \mathbb{Q} \longrightarrow \mathbb{Q}, (x)f = x + 1 \;\; \forall\, x \in \mathbb{Q}$.

Let $g : \mathbb{Q} \longrightarrow \mathbb{Q}, (x)g = x + 1/2 \;\; \forall\, x \in \mathbb{Q}$. Now $g \circ g = f \circ f$ but of course $f \neq g$ since $(0)f = 1/3 \neq 1/2 = (0)g$.

For further discussion on orderable groups see the text by Mura and Rhemtulla [14].

Now we give another corollary to theorem 3.1.

This corollary generalises theorem 2.6 in a way, since it strengthens the conclusion concerning the properties of G from being locally fresh to being fresh, but in return it no longer asserts that this fresh order on G is a well order.

Corollary 3.1.2. *All torsion-free abelian groups are fresh.*

Given theorem 3.1 this result follows directly from the fact that torsion-free abelian groups are right(left) orderable (see [14]).

We will conclude this chapter by making some observations related to theorem 3.1.

We focus on corollary 3.1 to theorem 3.1 being as we mentioned a possibly stronger statement than that of the theorem. The converse to the corollary is not true for the following reason.

The free group $F = < x,\ y | - >$ is fresh with respect to short-lex ordering $<$, but $x_1 g^{-1} x_2 < g \;\; (x_1,\ x_2 < g)$ is false. Take $x_1 = yx$; $x_2 = x^3$; $g = x^{-1}yx$ and the order of the extended alphabet $x < y < x^{-1} < y^{-1}$. Now we have $x_1 g^{-1} x_2 =$

$yxx^{-1}y^{-1}xx^3 = x^4 > x^{-1}yx = g.$

However, in the attempt to reverse corollary 3.1 we can assert the following.

Statement 3.1 (1). *If an ordering $<$ is fresh on a group G then $\forall g \in G$ and $\forall x_1,\ x_2 \in G$ such that $x_1,\ x_2 < g$ it follows that $x_1 g^{-1} x_2 \neq g$.*

Proof. We assume for contradiction that $x_1 g^{-1} x_2 = g$. We take an arbitrary $p_1 \in G$ and let $p_2 = p_1 g x_2^{-1}$. Now we have

$p_1 g = (p_1 g x_2^{-1}) x_2 = p_2 x_2,\ x_2 < g$

$p_2 g = p_1 g x_2^{-1} g = p_1 x_1,\ x_1 < g$

Consequently we have $Pg \subset \bigcup_{x<g} Px$, where $P = \{p_1,\ p_2\}$. This is a contradiction to $<$ being fresh on G.

We should also mention that under the conditions of corollary 3.1 the "symmetric" conclusion can be drawn. Symmetry here means that we "change" the definition of an ordering being fresh by multiplying on the left, not on the right. More precisely, the following is true.

Statement 3.1 (2). *If a group G has a linear ordering $<$ such that $x_1 g^{-1} x_2 < g\ \ \forall x_1,\ x_2 < g$ then*

$$gP \not\subseteq \bigcup_{x<g} xP$$

for any finite non-empty $P \subseteq G$ and $\forall g \in G$.

We know of two proofs of this statement. The first one is direct and is in fact essentially the same as that of theorem 3.1. All what is required to do in order to construct this proof is to revise the proofs of lemmas 2.1.1, 2.1.2 from chapter 2 and of theorem 3.1 making all necessary changes in accordance with "modified" definition of an ordering being fresh. The process is entirely analogous to that of those proofs and we are not going to repeat it here.

The second proof, which follows, relies upon the statement of theorem 3.1 which had already been proved.

Proof. From corollary 3.1 we know that $<$ is fresh on G. Thus G is torsion-free. We suppose for contradiction that $\exists P,\ g$ such that

$$gP \subseteq \bigcup_{x<g} xP$$

where $g \in G$ and P is a finite non-empty subset of G. We consider two sets $\Omega_1 = \{yP \mid y \in G\}$ and $\Omega_2 = \{Py \mid y \in G\}$ and a map $\alpha : \Omega_1 \longrightarrow \Omega_2,\ yP \mapsto Py\ \forall y \in G$.

The map α is bijective. Indeed, for any $Py \in \Omega_2\ \exists \omega = yP \in \Omega_1$ such that $(yP)\alpha = Py$ so α is surjective. It is also injective, since if $Py_1 = Py_2$ then we have $(\star)\quad y_1 = y_2$ and therefore $y_1P = y_2P$.

The justification of $(\star)$ is as follows. We have $Py_1y_2^{-1} = P$, so $Py = P$ where $y = y_1y_2^{-1}$. Thus for any $p_1 \in P\ \exists p_2 = p_1y$ and hence we have $p_1y^2 = p_2y = p_3 \in P$. We can iterate in this manner and since P is finite we obtain $p_1y^{k-1} = p_k = p_l = p_1y^{l-1}$ and $l < k$. Hence, $y^{k-1} = y^{l-1}$, so $y^{k-l} = 1$ where $k - l > 0$. Since G is torsion-free we conclude that $y = 1$. On the other hand $y = y_1y_2^{-1}$, so $y_1y_2^{-1} = 1$ which means $y_1 = y_2$.

Now, since $<$ is fresh on G we have

$$Pg \not\subseteq \bigcup_{x<g} Px.$$

We can now apply α since it is bijective to obtain

$$(Pg)\alpha \not\subseteq \bigcup_{x<g} (Px)\alpha.$$

But this is the same as

$$gP \not\subseteq \bigcup_{x<g} xP.$$

This contradicts our assumption. The statement is therefore proved.

Finally we will give a criterion of a linear ordering being fresh on a group G for subsets P of size 2.

Statement 3.1 (3). *Any linear ordering $<$ of a group G is fresh for all sets P of size 2 in G if and only if $\forall g \in G,\ \forall x \in G$ such that $x < g$ we have $gx^{-1}g > g$.*

Proof.

1) If for every $P \subset G$ of size 2 we have $Pg \not\subseteq \bigcup_{x<g} Px$ then $gx^{-1}g > g \quad \forall g \in G,\ \forall x \in G$ such that $x < g$.

Otherwise, if $gx^{-1}g = g$ we have $g = x < g$ which is absurd. If on the other hand $gx^{-1}g < g$ then we can let $x_2 = gx^{-1}g < g$ and pick $p_1 \in G$ arbitrarily. We also let $p_2 = p_1 g x^{-1}$. Now we have

$$p_1 g = p_1 g x^{-1} x = p_2 x \quad\quad x < g$$

$$p_2 g = p_1 g x^{-1} g = p_1 x_2 \quad x_2 < g.$$

Therefore we have $Pg \subseteq \bigcup_{x<g} Px$ for $P = \{p_1,\ p_2\}$ which violates the condition of the statement.

2) If $\forall g \in G,\ \forall x \in G$ such that $x < g$ we have $gx^{-1}g > g$ then for every $P \subset G$ of size 2 we have $Pg \not\subseteq \bigcup_{x<g} Px$.

Indeed, if $\exists g \in G,\ P \subset G$ such that $|P| = 2,\ Pg \subseteq \bigcup_{x<g} Px$ and also $gx^{-1}g > g$ for every $x \in G,\ x < g$ then $\exists x_1,\ x_2 < g$ in G such that

$$p_1 g = p_2 x_2$$

$$p_2 g = p_1 x_1.$$

In other words

$$p_2 = p_1 g x_2^{-1}$$

$$p_1 g x_2^{-1} g = p_1 x_1,$$

which means that $g x_2^{-1} g = x_1 < g$ which is a contradiction to our conditions.

The statement is now proved.

Obviously from this statement and corollary 3.1 the following observation on properties of groups can be easily derived. If a linear ordering $<$ on a group G is such that given an arbitrary element $g \in G$ we have $x_1 g^{-1} x_2 < g \ \ \forall x_1, x_2 < g$ then it follows that $g x^{-1} g > g \ \ \forall x < g$.

Also we can use statement 3.1 (3) to prove that if a linear ordering $<$ on a group G is fresh for any subset of G of size 2 then G is torsion-free.

Indeed, let a linear ordering $<$ of a group G be fresh for any subset of G of size 2. Now, by statement 3.1 (3) for any $g, \ x \ \in G$ such that $x < g$ we have $g x^{-1} g > g$. Therefore $(g x^{-1} g) g^{-1} (g x^{-1} g) > g x^{-1} g$ or in other words $g x^{-1} g x^{-1} g > g x^{-1} g > g > x$. Iterating we obtain

$$\ldots > (g x^{-1})^n g > \ldots > (g x^{-1})^1 g > (g x^{-1})^0 g > x.$$

If for some $n \in \mathbb{N}$ we had $(g x^{-1})^n = 1$ then this would force $g > g$ which is absurd. Thus for any $g, \ x \ \in G$ such that $x < g$ the element $\alpha = g x^{-1} \neq 1$ has infinite order.

Now, given any $\alpha \in G, \ \alpha \neq 1$ we have that either $\alpha = g x^{-1}$ for some $g, \ x \ \in G$ such that $x < g$, in which case it has infinite order, or for any $g, \ x \ \in G$ such that $x < g$ we have $\alpha \neq g x^{-1}$. We claim that in the latter case α also has infinite order.

Indeed, we have $\alpha x \leq x \quad \forall x \in G$. Otherwise there would be $\hat{x} \in G$ such that $\alpha\hat{x} > \hat{x}$. Now let $g = \alpha\hat{x}$ to obtain $\alpha = g\hat{x}^{-1}, \quad g > \hat{x}$ which contradicts the assumption. Also $\alpha x \neq x$ since $\alpha \neq 1$. Thus, in fact, $\alpha x < x \quad \forall x \in G$. Iterating we obtain

$$x > \alpha x > \alpha^2 x > \ldots > \alpha^n x > \ldots$$

which means that α has infinite order.

Thus $\forall \alpha \in G$ the order of α is infinite. So G is torsion-free.

Returning to the general situation of studying fresh groups we pose an important and very natural question.

We know that fresh groups are torsion-free, but is the converse true as well? In other words, is the set of torsion-free groups strictly wider than the set of fresh groups or do they coincide?

We do not know the answer to this question but we conjecture that all torsion-free groups are fresh (as we mentioned in the introduction).

If the conjecture could be shown to be true then the following result would immediately yield the result that a group algebra of a torsion-free group cannot have zero-divisors. This would be a remarkable result, and so if our conjecture is true, we suspect that it does not admit an elementary proof.

Theorem 3.2. *Let G be a freshly orderable group and let R be an integral domain. The group ring RG does not have zero-divisors.*

Proof. We shall assume for contradiction that there are zero-divisors λ and μ in $RG - \{0\}$, so $\lambda\mu = 0$.

Let $<$ be a fresh order of G. Let $P = \text{supp}(\lambda)$. In other words

$$P = \{g \mid \lambda = \sum_{g \in G}{}' r_g g,\ r_g \neq 0\}.$$

Let $Q = \text{supp}(\mu)$. We choose $q \in Q$ which is maximal with respect to $<$. Such q obviously exists since Q is finite.

Now, since $<$ is fresh in G, we have

$$Pq \not\subseteq \bigcup_{h \in G,\ h<q} Ph.$$

We will denote $T = \bigcup_{h \in G,\ h<q} Ph$.

Thus we have $(\star)$: $Pq \not\subseteq T$. Let RT be the R-submodule of RG generated by T.

We can write $\mu = \sum'_{h<q} r_h h + r_q q$, since $\mu = \sum'_{h \in G} r_h h$ where q is maximal with respect to $<$ among those elements h of μ whose coefficients are non-zero.

Now, $\lambda = \sum'_{p \in G} a_p p$ where $a_p \neq 0$ only for $p \in P$. Thus $\lambda = \sum_{p \in P} a_p p$.

Next we write $\lambda\mu = (\sum_{p \in P} a_p p)(\sum_{h \in Q,\ h<q} r_h h + r_q q) = \theta + \lambda r_q q$ where $\theta \in RT$. Indeed, when multiplying $p \in P$ by $h \in Q,\ h < q$ we get elements from $\bigcup_{h \in G,\ h<q} Ph$ or, in other words, from T, with the corresponding coefficients from R. Thus, indeed, their sum which we denoted by θ is in RT.

However, the element obtained by multiplying λ by $r_q q$ is not in RT because otherwise we would have that the group elements in the support of $\lambda r_q q$ are in T, but they also form Pq which forces a conclusion $Pq \subseteq T$ violating $(\star)$. Thus, indeed, $\lambda r_q q \notin RT$.

On the other hand we can derive the opposite statement to get a necessary contradiction.

Indeed, $\lambda\mu = 0$ and hence $\theta + \lambda r_q q = 0$, where $\theta \in RT$. Therefore $\lambda r_q q \in RT$.

Thus our assumption was wrong and so RG does not have zero-divisors. The theorem is now proved.

We note that a stronger result is true. More precisely, the same proof works to show that the group ring RG has no zero-divisors if G is locally fresh.

CHAPTER 4

In this chapter we tackle our group theory problems by the means of mathematical logic. This approach proves to be successful.

First, going back to our original problem of solving infinite systems of linear equations on groups, we will obtain the interesting result given below as theorem 4.1 (1) or equivalently as theorem 4.1 (2). To this end we shall suppose that the field k of values of the variables is finite of size p^n where p is a prime and $n \in \mathbb{N}$. We will write $k = \{\kappa_0, \kappa_1, \kappa_2, \dots, \kappa_{p^n-1}\}$ where $\kappa_i \neq \kappa_j$ if $i \neq j$ $(i,\ j \in \{0, \dots, p^n - 1\})$.

Theorem 4.1 (1). *Suppose that T is a system of templates for a group G with values in a field k. Let the system of equations generated be E. Suppose that E admits only the trivial solution. It follows that the fact that E admits only the trivial solution can be verified by examination of a finite subset of E.*

Rather than prove this theorem directly, we reduce it to a simpler theorem. The point is that if one can show that any single prescribed unknown x_g must vanish then one can immediately deduce that all unknowns x_h, $h \in G$ must vanish by symmetry. Thus we reduce to showing that if E admits only the trivial solution and we distinguish an unknown x_g^* then the fact that $x_g^* = 0$ can be deduced by examination of a finite subset of E.

It turns out to be convenient to twist the formulation round in the following harmless way.

Theorem 4.1 (2). *If for some $g_* \in G$ every finite collection α of equations has a solution $x_g = d_{g,\alpha}$ $\forall g \in G$ with $d_{g_*,\alpha} \neq 0$ then there is a simultaneous solution in k to the entire collection E of equations with the property that x_{g_*} does not vanish.*

The means by which we are going to prove the result is the Gödel-Mal'cev compactness theorem for first order languages of mathematical logic. We will use it in the following form:

If every finite subset P of a first order language Γ has a model then Γ has a model.

We recall that a language of mathematical logic consists of formal symbols and special formulas containing these symbols. A first order language allows arbitrary collections of so-called constant symbols, a countable collection of variable symbols, the usual logical connectives and a collection of function symbols and relation symbols including equalities. These symbols can be combined into well formed formulas (grammatically correct phrases) in obvious ways. The reason for the name *first order* is that quantification is only permitted over elements (and not over subsets). We refer a reader to the following literature on mathematical logic for more specific descriptions: Enderton [7], Hamilton [9], Miller [12], Novikov [16].

Probably the most well-known application of the compactness theorem is the proof of Mal'cev's result that if a group is locally linear then it is linear.

For the sake of explanation we will, without philosophical prejudice, assume the existence of a Platonic reality where ordinary mathematical objects (groups, fields etc) exist.

To say that a first order language Γ has a model means that there is a correspondence between the constant symbols of Γ and some elements of an object in this Platonic mathematical reality such that all the formally defined functions and relations of Γ are existent functions and relations of the corresponding objects of this mathematical theory and all the formal equations of Γ actually hold in the theory.

We will now construct a suitable first order language which will in fact mimic the algebraic situation under consideration. First of all we construct a first order theory of fields.

We introduce variable symbols $y,\ z,\ w$ together with constant symbols $b_0,\ b_1, \dots , b_{p^n-1}$ which will correspond to the field elements. In particular b_0 and b_1 will correspond respectively to 0 and 1 of the field. We also introduce the following formal statements which contain an infix function symbol "$\cdot$".

1) $\forall y\ (y \cdot b_1 = y)$

2) $\forall y\ ((y \neq b_0) \to (\exists z\ (y \cdot z = b_1))$

3) $\forall y\ \forall z\ \forall w\ ((y \cdot z) \cdot w = y \cdot (z \cdot w))$

4) $\forall y\ \forall z\ (y \cdot z = z \cdot y)$.

Thus we have mimicked the multiplicative group structure of a field into our first order language. Next we similarly mimic an additive group structure of a field by introducing the following formal statements which contain a function symbol "+".

1') $\forall y\ \exists z\ (y + z = b_0)$

2') $\forall y\ \forall z\ \forall w\ ((y + z) + w = y + (z + w))$

3') $\forall y\ \forall z\ (y + z = z + y)$

4') $\forall y\ (y + b_0 = y)$.

Now we complete encoding a description of a field in our first order language by mimicking the distributive law. We insist that

$$\forall y\ \forall z\ \forall w\ ((y + z) \cdot w = y \cdot w + z \cdot w). \qquad (\star)$$

Finally we reflect in our language the assertion that the field k is finite of size p^n.

We do that by insisting that

$$\forall y\ ((y = b_0) \vee (y = b_1) \vee \ldots \vee (y = b_{p^n-1})). \qquad (\star\star)$$

Now we let the constant symbols $b_0,\ b_1, \ldots, b_{p^n-1}$, the variable symbols $y,\ z,\ w$ and function symbols "$\cdot$" and "+" together with the formal statements (1), ... , (5), (1'), ... , (5'), ($\star$) and ($\star\star$) be a first order language $\hat{\Gamma}$.

We complete our language as follows.

To each element of our group G we introduce a constant symbol c_g. We also introduce a constant symbol a_{κ_i} to each element κ_i of the finite field k insisting that $a_{\kappa_i} \neq a_{\kappa_j}$ if $i \neq j$ $(i,\ j \in \{0, \ldots, p^n - 1\})$. For simplicity of notations we will write a_i instead of a_{κ_i} $(i, \in \{0, \ldots, p^n - 1\})$. We insist that a_0 corresponds to $0 \in k$.

Next we consider a Platonic collection of infinitely many equations in infinitely many unknowns with coefficients in the finite field k such that each unknown is labelled by a group element g and each equation has the following form.

$$\Sigma' \kappa_i x_g = 0. \qquad (*)$$

Here κ_i and g range respectively over a finite subset of k and G and therefore Σ' is a finite sum. We introduce to each equation (*) of this system a formal equation

$$\Sigma' a_i c_g = a_0$$

by introducing to each Platonic unknown x_g, $g \in G$ a constant symbol c_g in our first order language and to each coefficient $\kappa_i \in k$ the corresponding constant symbol a_i in our first order language. Next we distinguish a particular $g_* \in G$.

Now we let the constant symbols c_g $(g \in G)$, a_i $(i \in \{0, \ldots, p^n - 1\})$, the fact that $a_i \neq a_j$ if $i \neq j$ $(i,\ j \in \{0, \ldots, p^n - 1\})$, together with the infinite collection of the formal equations and a formal inequality $c_{g_*} \neq a_0$ be a language β. We note that β is a first order language.

Next we put $\Gamma = \hat{\Gamma} \cup \beta \cup \rho$, where ρ is a set of formal equations $a_0 = b_0,\ a_1 = b_1, \ldots, a_{p^n-1} = b_{p^n-1}$. Thus we have ensured that $1 \neq 0$ in a field and that the field elements $b_0,\ b_1, \ldots, b_{p^n-1}$ are all different since we insisted earlier on that $a_i \neq a_j$ if $i \neq j$. Thus we have constructed a first order language Γ.

We are now ready to prove theorem 4.1 (2).

Proof. We will apply the compactness theorem to Γ to deduce that our theorem 4.1 (2) is true. The hypothesis of theorem 4.1 (2) ensures that every finite collection α of equations of the system S has a solution in k of the form $x_g = d_g$ $(\forall g \in G)$ with $d_{g_*} \neq 0$. This means that every finite subset P of our first order language Γ has a model.

Indeed, given our group G, the field k and finite set of equations α we can place every constant symbol c_g $(g \in G)$ in correspondence with $d_{g,\alpha} \in k$ and every constant symbol a_i $(i \in \{0, \ldots, p^n - 1\})$ in correspondence with a field element $\kappa_i \in k$. We obtain that any finite collection of equations $\Sigma' \kappa_i d_{g,\alpha} = 0$ holds in k together with $d_{g_*,\alpha} \neq 0$. Also, conditions (1), ... , (5), (1'), ... , (5'), ($\star$) and ($\star\star$) obviously hold for $y,\ z,\ w \in k$ since k is a field and has size p^n.

Thus we have that the language $\hat{\Gamma}$ together with the set of equations ρ and any finite subset of β has a model. This implies that any finite subset P of Γ has a model.

Thus we have checked that the hypothesis of the compactness theorem holds and therefore we conclude that Γ has a model. This means that, first, there is a correspondence between every constant symbol a_i ($i \in \{0, \ldots, p^n - 1\}$) and an element κ_i of a certain algebraic structure $\hat{k}$. By (1), ... , (5), (1'), ... , (5'), ($\star$) and ($\star\star$) this structure $\hat{k}$ is forced to be a field of size p^n with a_0 and a_1 in correspondence with its 0 and 1. The fact that $|\hat{k}| = p^n$ means that $\hat{k}$ is isomorphic to our field k.

Secondly, there is a correspondence between every constant symbol c_g ($g \in G$) and some field element $d_g \in \hat{k} \cong k$ such that all the equations $\Sigma' a_i c_g = a_0$ become valid equalities $\Sigma' \kappa_i d_g = 0$ in the field together with $d_{g_*} \neq 0$. But this means precisely that our system S has a non-trivial solution in the field k with the property $x_{g_*} \neq 0$.

The theorem is now proved.

Next we will see that the same method of mathematical logic also works when dealing with questions of fresh orderings of groups.

Theorem 4.2. *If every finitely generated subgroup of G is freshly orderable then G is freshly orderable.*

We construct a first order language as follows.

To each group element g we introduce a constant symbol c_g. We insist that these constant symbols be distinct so if g and h are distinct group elements then $c_g \neq c_h$.

We introduce a product on these constant symbols which mimics the multiplicative structure of G. Thus we have $c_g \cdot c_h = c_{gh}$ for every pair of group elements $g,\ h \in G$.

Next we mimic the group axioms by introducing variable symbols $y,\ z,\ w$ and insisting that

1) $\forall y\ (y \cdot c_1 = c_1 \cdot y = y)$

2) $\forall y\ \exists z\ (y \cdot z = z \cdot y = c_1)$

3) $\forall y\ \forall z\ \forall w\ ((y \cdot z) \cdot w = y \cdot (z \cdot w))$.

Now we introduce a binary relation symbol $\ltimes$ which mimics a linear order on G. We do this by insisting that

1') $\forall y\ \forall z\ ((y \ltimes z) \to \neg(z \ltimes y))$

2') $\forall y\ \forall z\ \forall w\ (((y \ltimes z) \wedge (z \ltimes w)) \to (y \ltimes w))$

3') $\forall y\ \neg(y \ltimes y)$.

Let $\hat{\Gamma}$ denote the first order language consisting of our constant symbols, the fact that they are distinct, our variable symbols, a relation symbol $\ltimes$, an infix function symbol "$\cdot$" and sentences (1), (2), (3), (1'), (2'), (3'). Thus $\hat{\Gamma}$ mimics a group structure of G and a linear order on G.

Next we will mimic the fact that this linear order on G is also a fresh order. We consider the following statements $A_n(a_1, \dots, a_n)$:

$$\forall y \forall z (z \ltimes y \to (a_1 y \neq a_1 z \wedge a_1 y \neq a_2 z \wedge \ldots \wedge a_1 y \neq a_n z)$$

$$\vee (a_2 y \neq a_1 z \wedge a_2 y \neq a_2 z \wedge \ldots \wedge a_2 y \neq a_n z)$$

$$\ldots$$

$$\vee (a_n y \neq a_1 z \wedge a_n y \neq a_2 z \wedge \ldots \wedge a_n y \neq a_n z)$$

Now we put $\Gamma = \hat{\Gamma} \cup \{A_n(c_{g_1}, \dots, c_{g_n}) \mid n \geq 1,\ (g_1, \dots, g_n) \in G^n\}$.

We are now ready to prove theorem 4.2.

Proof. We apply the compactness theorem for first order languages to Γ.

By the hypothesis of theorem 4.2 we have that every finitely generated subgroup of G is freshly orderable. This enables us to deduce that every finite subset P of our first order language Γ has a model.

Indeed, to each element $g \in G$ there corresponds a constant symbol c_g. Now, since G is a group all the formal sentences of $\hat{\Gamma}$ containing constant symbols c_g which mimic a group structure become true for the collection of all $g \in G$.

Also, every sentence $A_n(c_{g_1}, \dots, c_{g_n})$ contains only finite number of constant symbols. Therefore a finite collection Ω of sentences $A_n(c_{g_1}, \dots, c_{g_n})$ also contains a finite collection of constant symbols, not necessarily all different. We consider a subgroup H_Ω generated by those group elements of G which correspond to the constant symbols involved in Ω. Thus H_Ω is finitely generated and therefore freshly orderable. We take a fresh order $\sqsubset_\Omega$ of H_Ω and place it in correspondence with a relation symbol $\ltimes_\Omega$. Since $\sqsubset_\Omega$ is linear it is obvious that (1'), (2') and (3') hold for $\sqsubset_\Omega$ with $y,\ z,\ w \in H_\Omega$. Also, since $\sqsubset_\Omega$ is fresh on H_Ω then for every finite non-empty $M \subset G$ and for every $y \in H_\Omega$ we have

$$My \not\subseteq \bigcup_{z \sqsubset_\Omega y} Mz.$$

This means that $\exists \hat{m} \in M$ such that $\hat{m}y \neq mz\ \forall m \in M$. In other words we have that $M = \{g_1, \dots, g_n\}$ corresponds to $\{c_{g_1}, \dots, c_{g_n}\}$ and the finite collection of sentences $A_n(g_1, \dots, g_n)$, which corresponds to Ω, holds.

Thus we obtained that the language $\hat{\Gamma}$ together with any finite collection of the sentences $A_n(c_{g_1}, \dots, c_{g_n})$ has a model. This implies that any finite subset P of Γ has a model.

Thus we have checked that the hypothesis of the compactness theorem holds and therefore we conclude that Γ has a model. This means that, first, there is a correspondence between the collection of constant symbols c_g $(g \in G)$ and an algebraic structure $\hat{G}$. The latter is forced to be a group by those statements of $\hat{\Gamma}$ which mimic a group structure.

Now, since $h \neq g$ in G forces $c_h \neq c_g$ we have that $\hat{h} \neq \hat{g}$ where $\hat{h},\ \hat{g}$ are elements of $\hat{G}$ corresponding to $c_h,\ c_g$. Thus we, in fact, have an injective homomorphism $g \rightarrow \hat{g}$ from G to $\hat{G}$ since $c_g \cdot c_h = c_{gh}\ \forall g,\ h \in G$. Now the image N of this homomorphism is a subgroup of $\hat{G}$ and $N \cong G$. Secondly, there is a binary relation $\sqsubset$ on $\hat{G}$, which corresponds to $\ltimes$, such that it satisfies (1'), (2'), (3') and the collection of statements $\{A_n(g_1, \cdots, g_n) \mid n \geq 1,\ (g_1, \cdots, g_n) \in G^n\}$. This means that $\forall P \subset G\ \ \forall g \in \hat{G}$ where P is non-empty and finite we have $Pg \not\subseteq \bigcup_{h \sqsubset g,\ h \in \hat{G}} Ph$. Hence $\forall P \subset G\ \ \forall g \in G$ where P is non-empty and finite we have $Pg \not\subseteq \bigcup_{h \sqsubset g,\ h \in G} Ph$. So, $\sqsubset$ is a fresh order on G.

Thus we have shown the existence of a fresh ordering $\sqsubset$ of G. The theorem is now proved.

Notice that this theorem says nothing about well fresh orderings, since the notion of a well order is not first order.

Finally we again use the compactness theorem to prove a theorem which we shall be exploring further in chapter 5. Proofs using the compactness theorem are somewhat repetitive when written out in full detail, and we think that the reader will now be sufficiently expert in the technique to allow an outline proof of this result.

Definition. *A group G is locally fresh in the subset sense if whenever P is a finite* *non-empty subset of G then there exists a linear order $<^P$ of the underlying set of G which is fresh for P and for any of its non-empty subsets. In other words, for every $g \in G$ and for every non-empty $L \subseteq P$ we have*

$$Lg \not\subseteq \bigcup_{h <^P g} Lh.$$

We say that $<^P$ is an ordering which is P-fresh in the subset sense on G.

Theorem 4.3. *A group G is freshly orderable if and only if G is locally freshly orderable in the subset sense.*

Outline proof. If G is fresh then clearly G is locally freshly orderable in the subset sense.

The converse is not quite so trivial. We construct a first order language by first writing down the group axioms, introducing a function symbol for multiplication. Then we encode G in the language by introducing distinct constant symbols c_g for each group element $g \in G$. Next we introduce a relation $<$ and the axioms which force it to be a linear order. This is all first order material.

Now for each finite subset P of G we introduce a sentence $\mathfrak{S}_P$ which asserts that $<$ is a fresh order for P and all of its subsets. Each of these sentences is first order.

Now we apply the compactness theorem. Any finite subset of the language involves only finitely many sentences of the form $\mathfrak{S}_P$. Let Q be the finite set which is the union of these $P's$. Now by hypothesis G equipped with the order $<^Q$ is a model of our finite subset of the language.

We deduce that the entire language has a model $\overline{G}$ equipped with a linear ordering $<'$. Here $\overline{G}$ is a group containing a copy of G as a subgroup (via the constant symbols). Now $<'$ restricted to G is a fresh ordering by the design of the first order language and we are done.

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CHAPTER 5

As we demonstrated in chapter 2 the property of groups of being freshly orderable is preserved by the formation of Cartesian products. This result was formulated as theorem 2.1 and was provided with two alternative proofs. The question arises whether the property of being locally fresh is also preserved by the formation of Cartesian products. In other words, whether the "locally fresh" version of theorem 2.1 is true.

Our attempts to prove this statement by modifying the proofs of theorem 2.1 or otherwise faced some genuine difficulties. However, in this connection we obtained a number of related results which we present in the current chapter.

At the end of chapter 3 we stated a conjecture that every torsion-free group is freshly orderable. We point out that if the "locally fresh" version of theorem 2.1 is false then the conjecture is also false. In other words, if the conjecture is true then it forces the sets of torsion-free groups, freshly orderable groups and locally freshly orderable groups to be the same sets. In this respect proving the conjecture correct would make the considerations in this chapter obsolete and the results trivially true.

However, our current results may shed some light on the nature of locally fresh orderings on groups.

We will start by introducing some notation and making some definitions.

Let $(G_i)_{i\in I}$ be a family of groups indexed by the set I. The Cartesian product of these groups has underlying set

$$\mathrm{Cart}_{i\in I}G_i = \{f \mid f : I \longrightarrow \bigcup_i G_i,\ (i)f \in G_i\ \forall i \in I\}.$$

The group law is given pointwise. If $f_1, f_2 \in \mathrm{Cart}_{i\in I}G_i$ then $f_1 f_2 : I \longrightarrow \bigcup_i G_i$,

$(i)(f_1 f_2) = (i)f_1(i)f_2 \ \forall i \in I$.

We have natural epimorphisms $\pi_i : \mathrm{Cart}_{i \in I} G_i \longrightarrow G_i$ given by $(f)\pi_i = (i)f$.

Sometimes, to simplify our notation we will write $(f)\pi_i$ as $\bar{f}$.

We denote by $<_i$ a linear ordering of G_i. We invoke the well ordering principle to obtain a well order $<_I$ on the indexing set I. Now we describe the following order $<$ on G.

If $g, h \in G$ we write $g < h$ if and only if $\exists j \in I$ such that $(j)g <_j (j)h$ but $(i)g = (i)h \ \forall i <_I j$.

Definition. *We will call such an ordering a natural ordering of G.*

Definition. *Let us suppose that G_i is a locally freshly orderable group $\forall i \in I$. Given a finite non-empty subset P of G we will call a natural ordering of G an individual $P-$order on G if and only if $<_i$ is fresh for $(P)\pi_i$ $\forall i \in I$.*

It is easy to see that $<$ is an ordering of G. In fact $<$ is a linear ordering. This is because if $g \neq h$ then $\Omega = \{i \mid i \in I, \ (i)g \neq (i)h\}$ is a non-empty subset of the well ordered set I. Let j be the minimal element of Ω and we are done.

Next, we state the following simple result.

Theorem 5.1 (1). *Let $(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. Let P be a finite non-empty subset of G with the property that $\exists j \in I$ such that $|(P)\pi_j| = |P|$. Let $\sqsubset$ be a natural ordering of G such that $<_j$ is fresh for $(P)\pi_j$ and $j = min_{<_I} I$. It follows that $\sqsubset$ is fresh for P on G.*

Proof. Let $|P| = n$. We assume for contradiction that $\sqsubset$ is not fresh for P on G.

This means that $\exists g \in G$ such that

$$Pg \subseteq \bigcup_{h \sqsubset g} Ph.$$

It follows that there is a system of n equations $p_i g = p_{s_i} h_i$ where $h_i \sqsubset g$ $\forall i \in \{1, \dots, n\}$ and $s_i \in \{1, \dots, n\}$.

Next we apply the map π_j to our equations. Suppose $(g)\pi_j = (h_i)\pi_j$ (for contradiction). Now it follows that $(p_i)\pi_j = (p_{s_i})\pi_j$ but π_j is injective when restricted to P so $i = s_i$ and thus $h_i = g$. However, $h_i \sqsubset g$ so our contradiction is established. Thus $(g)\pi_j \neq (h_i)\pi_j$ $\forall i \in \{1, \dots, n\}$. This now violates the freshness of $<_j$ for $(P)\pi_j$ on G_j and we are done.

The theorem is now proved.

Now it will be easy to see that in certain restricted circumstances we can generalize theorem 5.1 (1) by relaxing the condition that j be the minimal element of the indexing set I with respect to the order $<_I$. We obtain the following.

Theorem 5.1 (2). *Let$(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. Let P be a finite non-empty subset of G with the property that $\exists j \in I$ such that $|(P)\pi_j| = |P|$. Let $\sqsubset$ be a natural ordering of G such that $<_j$ is fresh for $(P)\pi_j$ and $|(P)\pi_k| = 1$ $\forall k <_I j$. It follows that $\sqsubset$ is fresh for P on G.*

Proof. We simply modify the proof of theorem 5.1 (1) in the obvious way.

Next we will introduce the following lemma which is used repeatedly to prove results in the sequel.

Lemma 5.2 (1). *Let $(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. Select a well order $<_I$ of the indexing set I. Let*

$j = min_{<_I} I$. Let the fibres of π_j intersected with P be $N_1, N_2, \dots, N_s$ and we arrange matters so that $|N_l| > 1 \Leftrightarrow l \leq t$. We insist that $t \geq 1$. Let $\sqsubset$ be a natural ordering of G with $<_I$ being our chosen order of I and such that $<_j$ is fresh for $(P)\pi_j$. If $\sqsubset$ is not fresh for P on G it follows that $\exists k \in \{1, \dots, t\}$ such that $\sqsubset$ is not fresh for N_k.

Proof. Since $\sqsubset$ is not fresh for P there is some $g \in G$ such that

$$Pg \subseteq \bigcup_{h \sqsubset g} Ph.$$

Let $\bigcup_{i=1}^{t} N_i = \{p_u \mid 1 \leq u \leq m\}$ and suppose $P = \{p_u \mid 1 \leq u \leq n\}$ where $|P| = n$. Now we know that

$$p_1 g = p_{i_1} h_1$$

$$p_2 g = p_{i_2} h_2$$

$$\dots$$

$$p_n g = p_{i_n} h_n,$$

where $h_u \sqsubset g \; \forall u \in \{1, \dots, n\}$. We shall write $\overline{x}$ for $(x)\pi_j$ whenever $x \in G$. Notice that $j = \min(I)$ forces $\overline{h_u} \leq_j \overline{g} \; \forall u \in \{1, \dots, n\}$.

Suppose that $v > m$ and (for contradiction) that $\overline{h_v} = \overline{g}$; it follows that $\overline{p_v} = \overline{p_{i_v}}$ but $v \neq i_v$ since $h_v \neq g$. This is impossible since $v > m$. We conclude that $\overline{h_v} < \overline{g}$ for $v > m$.

Now suppose (for contradiction) that for every $r \in \{1, \dots, t\}$ there exists $p_{j_r} \in N_r$ with $\overline{h_{j_r}} < \overline{g}$. Now

$$\overline{P} = \overline{\{p_{j_r} \mid 1 \leq r \leq t\} \cup \{p_v \mid m < v \leq n\}},$$

so $\overline{P\bar{g}} \subseteq (\bigcup_{r=1}^{t} \overline{Ph_{j_r}}) \cup (\bigcup_{v=m+1}^{n} \overline{Ph_v}) \subseteq \bigcup_{\bar{h}<_j \bar{g}} \overline{Ph}$.

This is the required contradiction (by freshness of $<_j$) so we can conclude that there is some $w \in \{1, \dots, t\}$ such that if $p_u \in N_w$ then $\overline{h_u} = \bar{g}$. Without loss of generality we may assume that $w = 1$. Now if $p_u \in N_1$ then $p_u g = p_{i_u} h_u$ with $\bar{g} = \overline{h_u}$, so $\overline{p_u} = \overline{p_{i_u}}$, so $p_{i_u} \in N_1$. Let $n_1 = |N_1|$.

Thus

$$N_1 g = \{p_{i_u} h_u \mid 1 \leq u \leq n_1\} \subseteq \bigcup_{u=1}^{n_1} N_1 h_u \subseteq \bigcup_{h \sqsubset g} N_1 h.$$

Thus $\sqsubset$ is not fresh for N_1.

The lemma is now proved.

Having proved the lemma it will be now transparent that in fact we can relax its conditions as follows.

Lemma 5.2 (2). *We may relax the condition of lemma 5.2 (1) that $j = min_{<_I} I$ provided we insist that $Im(\pi) = 1$ $\forall i < j$.*

Proof. Simply make minor adjustments to the proof of lemma 5.2 (1).

We will now use these results to prove the following.

Theorem 5.2. *Let $(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. Suppose that P is a finite non-empty subset of G. Suppose $N \subseteq P$, $|N| > 1$ and that N enjoys both of the following properties:*

(1) $\exists j \in I$ such that $(N)\pi_j = 1$, but π_j is injective when restricted to $P - N$.

(2) $\forall i \in I - \{j\}$ we have

either $|(N)\pi_i| = |(P)\pi_i| = |N|$

or $(N)\pi_i = 1$ but π_i is injective when restricted to $P - N$.

Select a well order $<_I$ of the indexing set I so that $j = min_{<_I} I$. It follows that any individual P-order (see the definition on page 88) $\sqsubset$ with $<_I$ being our chosen order of I is fresh for P on G.

Proof. We assume for contradiction that $\sqsubset$ is not fresh for P on G. As it is easy to see it follows from (1) that lemma 5.2 (1) now applies. We can therefore conclude that $\sqsubset$ is not fresh for N on G.

However, theorem 5.1 (2) yields that $\sqsubset$ is fresh for N because, unless $\forall i \in I$ we have $|(P)\pi_i| = |P| - |N| + 1$, $|(N)\pi_i| = 1$ there is some $l \in I$ such that $|(N)\pi_l| = |(P)\pi_l| = |N|$. We now take $\hat{l} = \min_{<_I} \Omega$, where Ω is the set of all such l and apply theorem 5.1 (2) using $\hat{l}$ as j in the theorem and using N as P in the theorem.

Also the case where $\forall i \in I$ we have $|(P)\pi_i| = |P| - |N| + 1$, $|(N)\pi_i| = 1$ is impossible because it means that $|N| = 1$ which violates the condition of N being not a singleton set. Thus the proof is now completed.

We continue to consider a group $G = \mathrm{Cart}_{i \in I} G_i$, where G_i is locally fresh $\forall i \in I$. The next lemma together with lemma 5.2 (1) makes it possible to prove the existence of the orderings fresh for some particular subsets P of G if the existence of a fresh order for their projections on a "smaller" Cartesian product is known. More precisely, we have the following result.

Lemma 5.3. *Let $(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. For some $j \in I$ let α_j be an epimorphism $G \longrightarrow G' = Cart_{i \in I - \{j\}}$ given by $(f)\alpha_j = f|_{I - \{i\}}$ $\forall f \in G$. Let π_i be as usual a natural homomorphism $G \longrightarrow G_i$ given by $(f)\pi_i = (i)f$. Let a finite non-empty subset P of G*

have the following properties: $\exists j \in I$ *such that*

1) $|(P)\pi_j| = 1$

2) there exists a natural ordering $\sqsubset_{G'}$ *on* G' *fresh for* $(P)\alpha_j$.

It follows that the following order $\sqsubset$ *of* G *is fresh for* P :

$<_I$ *is a well order of* I *constructed by joining the element* j *to the well order* $<_{I-\{j\}}$ *associated with* G' *so that* $j = min_{<_I} I$.

$<_i$ *is the same ordering of* G_i *as that in* $\sqsubset_{G'}$ *for any* $i \in I - \{j\}$.

$<_j$ *is a linear ordering of* G_j.

Proof. We assume for contradiction that $\sqsubset$ is not P-fresh. This means that for some $g \in G$ we have

$$Pg \subseteq \bigcup_{h \sqsubset g} Ph.$$

It follows that we have a system of equations

$$p_u g = p_{k_u} h_u \quad (1 \leq u \leq n) \tag{$\star$}$$

where $|P| = n$, $P = \{p_u \mid 1 \leq u \leq n\}$, $k_u \in \{1, \dots, n\}$, $\forall u \in \{1, \dots, n\}$ and $h_u \sqsubset g$ $\forall u \in \{1, \dots, n\}$.

Now, when $h \sqsubset g$ it follows that

either $(j)h <_j (j)g$

or both $(j)h = (j)g$ and $(h)\alpha_j \sqsubset_{G'} (g)\alpha_j$

Now $|(P)\pi_j| = 1$ so $(p_u)\pi_j$ is independent of u, so $(h_u)\pi_j = (g)\pi_j$ $\forall u \in \{1, \dots, n\}$. We now apply α_j to our system of equations $(\star)$ to obtain

$$(p_u)\alpha_j (g)\alpha_j = (p_{k_u})\alpha_j (h_u)\alpha_j$$

for all $u \in \{1, \ldots, n\}$ and with $(h_u)\alpha_j \sqsubset_{G'} (g)\alpha_j$ $\forall u \in \{1, \ldots, n\}$.

Thus

$$(P)\alpha_j(g)\alpha_j \subseteq \bigcup_{x \sqsubset_{G'} (g)\alpha_j} (P)\alpha_j x$$

which contradicts the assumption that $\sqsubset_{G'}$ is a fresh ordering for $(P)\alpha_j$ on G' and we are done.

Now we are ready to prove that a Cartesian product of locally fresh groups has orderings which are fresh for subsets of small sizes. More precisely, we have the following theorem.

Theorem 5.3. *Let $(G_i)_{i \in I}$ be a family of locally freshly orderable groups and suppose that $G = Cart_{i \in I} G_i$. If a subset P of G is of size at most 4, then there exists an ordering of G which is fresh for P.*

Proof. We note that as usual $\forall i \in I$ we have a natural homomorphism $\pi_i : G \longrightarrow G_i$ given by $f \to (i)f$ $\forall f \in G$. For any given i the homomorphism π_i is not necessarily injective. So, if $P \subset G$ is of size n then we have $|(P)\pi_i| \leq |P|$ with $((|(P)\pi_i| = |P|) \Leftrightarrow (\pi_i|_P \text{ is injective}))$.

Let the fibres of $\pi_i|_P$ be M_u $(1 \leq u \leq k)$ so $\bigcup_{u=1}^{k} M_u = P$ and if $a \in (P)\pi_i$ then $(a)\pi_i^{-1} \cap P = M_{v_a}$ for some $v_a \in \{1, \ldots, k\}$. If $|(P)\pi_i| < |P|$ then $\exists u$ such that $|M_u| > 1$. If $|(P)\pi_i| = |P|$ then $|M_u| = 1$ $\forall u \in \{1, \ldots, k\}$.

Thus the situation in G_i with $(P)\pi_i$ in relation to P $\forall i \in I$ generally depends first of all on the possible partitions of the size of $|P|$. Having made this observation we will now proceed with the proof.

1) Let $|P| = 1$. Obviously in this case any linear ordering $\sqsubset$ of G will be fresh

for P. Indeed, if having $P = \{p_1\}$ we suppose that for some $g \in G$ we have

$$p_1 g \subseteq \bigcup_{h \sqsubset g} p_1 h$$

then it follows that $g \subseteq \bigcup_{h \sqsubset g} h$. This means that $g \sqsubset g$ which is absurd.

2) Let $|P| = 2$. In other words $P = \{p_1, p_2\}$. We have the following partitions of 2: $2 = 1 + 1 = 2 + 0$. Therefore for every $i \in I$ we have either $|(P)\pi_i| = 2 = |P|$ or $|(P)\pi_i| = 1$. Thus, unless $|(P)\pi_i| = 1 \ \forall i \in I$ we can apply theorem 5.1 (2) to conclude that a particular natural ordering described there is fresh for P. We note that in particular an individual P-order is fresh for P on G.

The case $|(P)\pi_i| = 1 \ \forall i \in I$ is impossible since it means that $|P| = 1$ whereas $|P| = 2 \neq 1$.

Thus we proved the theorem for P of size 2.

3) Let $|P| = 3$. In other words $P = \{p_1, p_2, p_3\}$. The partitions of 3 are as follows: $3 = 1 + 1 + 1 = 2 + 1 = 3 + 0$. Therefore we consider the following cases. First of all, if $\exists j \in I$ such that $|(P)\pi_j| = |P| = 3$ then we can apply theorem 5.1 (1) (or, if you like, theorem 5.1 (2) which is a broader version of the former). We conclude there is an ordering of G described in the above results which is fresh for P and we are done. We note that this ordering is a natural one.

Thus without loss of generality we may now assume that $\forall i \in I$ we have $|(P)\pi_j| < |P| = 3$. This means that $\forall i \in I$ we have

either $\exists N \subsetneq P, |N| = 2$ such that $|(N)\pi_i| = 1, \ |(P)\pi_i| = 2$;

or $|(P)\pi_i| = 1$.

The first case definitely occurs for some $i \in I$ because otherwise we just have $|(P)\pi_i| = 1 \ \forall i \in I$ which means that $|P| = 1$ whereas $|P| = 3 \neq 1$.

Thus for some $j \in I$ we have $\exists N \subsetneq P,\ |N| = 2,\ |(N)\pi_j| = 1$.

Let α_j be an epimorphism $G \longrightarrow G' = \mathrm{Cart}_{i \in I - \{j\}}$ given by $(f)\alpha_j = f|_{I-\{i\}}$ $\forall f \in G$. We consider $(N)\alpha_j \subset G'$. Since $|(N)\alpha_j| = |N| = 2$ we can apply the previous result of this theorem which we proved above. It says that for such subset $(N)\alpha_j$ of size 2 of the Cartesian product G' of locally freshly orderable groups there exists an ordering $\sqsubset_{G'}$ of G' which is fresh for $(N)\alpha_j$. Thus putting this together with the condition that $|(N)\pi_j| = 1$ we can apply lemma 5.3.

We conclude that an order $\sqsubset$ described in the lemma is fresh for N on G. This ordering $\sqsubset$ is a natural ordering of G and it has $<_I$ such that j is the minimal element of I with respect to $<_I$. Also $\sqsubset$ is such that $<_j$ is a linear ordering of G_j. In particular if $<_j$ is fresh for $(P)\pi_j$ we will denote such an ordering $\sqsubset$ by $\sqsubset'$.

Thus we have that $\sqsubset'$ is fresh for N on G. We claim that $\sqsubset'$ is also fresh for P on G. Indeed, if we assume for contradiction that $\sqsubset'$ is not P-fresh on G then by lemma 5.2 (1) we obtain that $\sqsubset'$ is not N-fresh on G. This contradicts our previous conclusion that $\sqsubset'$ is fresh for N on G.

Thus the assumption was wrong and so we have that any ordering $\sqsubset'$ on G is fresh for P. We note that $\sqsubset'$ is a natural ordering of G.

4) Let $|P| = 4$. In other words $P = \{p_1, p_2, p_3, p_4\}$. The partitions of 4 are as follows: $4 = 1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4 + 0$. Therefore we consider the following cases.

First of all, if $\exists j \in I$ such that $|(P)\pi_j| = |P| = 4$ then we apply theorem 5.1 (1) (or theorem 5.1 (2)) to conclude that there is an ordering of G which is fresh for P.

Thus without loss of generality we may now assume that $\forall i \in I$ we have $|(P)\pi_i| < |P| = 4$. This means that there are the following possibilities.

(a) $\exists j \in I$ such that $\exists N \subsetneq G,\ |N| = 3,\ |(N)\pi_j| = 1,\ |(P)\pi_j| = 2$.

In this case we can apply lemma 5.3. Indeed, we have a set $N \subset G$ such that $|(N)\pi_j| = 1$. Also, since $|N| = 3$ then $|(N)\alpha_j| = 3$ where α_j is an epimorphism $G \longrightarrow G' = \mathrm{Cart}_{i\in\{I-j\}}G_i$ given by $(f)\alpha_j = f|_{G'}\ \forall f \in G$. Now, by part 3 of this theorem, since G_i is locally freshly orderable $\forall i \in I$, there is a natural ordering of G' which is fresh for $(N)\alpha_j$. We will denote it by $\sqsubset_{G'}$.

We have now justified that lemma 5.3 applies. Thus we conclude that there is an ordering $\sqsubset$ on G which is fresh for N. We note that it follows from lemma 5.3 that $\sqsubset$ is a natural ordering of G with $j = \min_{<_I} I$ and $<_j$ is a linear ordering of G_j. In particular, if an ordering $\sqsubset$ has $<_j$ being fresh for $(P)\pi_j$ we will denote it by $\sqsubset'$.

Thus we obtained (*): $\sqsubset'$ is N-fresh on G. We claim that $\sqsubset'$ is also fresh for P on G. Indeed, if we assume that $\sqsubset'$ is not P-fresh we can apply lemma 5.2 (1) to conclude that $\sqsubset'$ is also not N-fresh. This contradicts our result (*) above.

So, our assumption was wrong and $\sqsubset'$ is P-fresh on G. Thus in the conditions (a) we have found a natural ordering of G which is P-fresh.

(b) $\exists j \in I$ such that $\exists N \subsetneq G,\ |N| = 2,\ |(N)\pi_j| = 1,\ |(P)\pi_j| = 3$.

It is easy to see that this case is in principle the same as (a) in terms of the proof strategy. We make necessary changes to the proof of (a) given above to conclude that there exists a natural ordering $\sqsubset'$ on G which is P-fresh.

Now we may assume without loss of generality that none of the cases described above occurs. Thus we have the final option which is as follows: $\forall i \in I$ we have

either (1): $\exists N_1^i,\ N_2^i \subsetneq P,\ |N_1^i| = |N_2^i| = 2,\ |(N_1^i)\pi_i| = |(N_2^i)\pi_i| = 1,\ |(P)\pi_j| = 2$

or (2): $|(P)\pi_i| = 1$.

Clearly, (1) occurs for some $j \in I$. Otherwise we would have $|(P)\pi_i| = 1 \ \forall i \in I$ which means $|P| = 1$ whereas $|P| = 4 \neq 1$.

Next we let $\sqsubset$ be an individual P-order of G such that $<_I$ has j as its minimal element. This means that $\sqsubset$ is a natural ordering of G with $<_I$ being a well order of the indexing set I such that $j = \min_{<_I} I$ and $<_i$ being fresh for $(P)\pi_i \ \forall i \in I$. We claim that $\sqsubset$ is P-fresh on G. We assume for contradiction that it is not so.

Thus we are in the position to apply lemma 5.2 (1) to conclude that $\sqsubset$ is not fresh either for N_1^j or for N_2^j. Without loss of generality we will suppose (**): $\sqsubset$ is not fresh for N_1^j.

On the other hand since $|N_1^j| = 2$ we have $\forall i \in I$

either $|(N_1^j)\pi_i| = 1$

or $|(N_1^j)\pi_i| = 2$.

If $|(N_1^j)\pi_i| = 1$ for some $i \in I$ then, of course, $<_i$ is fresh for $(N_1^j)\pi_i$. If $|(N_1^j)\pi_i| = 2$ for some $i \in I$ then we have $2 = |(N_1^j)\pi_i| \leq |(P)\pi_i| \leq 2$ which means that $|(N_1^j)\pi_i| = |(P)\pi_i|$. In fact it means that $(N_1^j)\pi_i = (P)\pi_i$ since we also have $(N_1^j)\pi_i \subset (P)\pi_i$. Therefore $<_i$ which is fresh for $(P)\pi_i$ is in fact fresh for $(N_1^j)\pi_i$.

Thus we have shown that $\sqsubset$ is in fact an individual N_1^j-order. Therefore, since $|N_1^j| = 2$ then by the part 2 of this theorem we conclude that $\sqsubset$ is fresh for N_1^j. This contradicts the result (**) above. Thus our assumption was wrong and so $\sqsubset$ is P-fresh.

We have now considered all possible cases and proved that there is always an ordering of G fresh for a subset P of a size n, where $n \in \{1, \ 2, \ 3, \ 4\}$. The theorem is therefore proved.

When a size of a subset P increases, so does the diversity of possible situations with $(P)\pi_i$ on G_i $\forall i \in I$. In this case our technique on its own seems to be not enough to reach the necessary conclusion concerning existence of a P-fresh ordering on G.

However, we will demonstrate how it does work in some special cases of subsets P of an arbitrary size. This is formulated in the following theorem which has theorem 5.1 (1) proved earlier as its particular case.

Theorem 5.4. *Let $\{G_i\}_{i\in I}$ be a family of locally freshly orderable groups with the indexing set I. Let $G = Cart_{i\in I}G_i$ be a Cartesian product of $\{G_i\}_{i\in I}$. It follows that for every non-empty subset P of G with the following properties there exists an ordering of G which is P-fresh.*

The properties of P are as follows: There exists a chain of nested subsets of P

$$N_r \subsetneq \ldots \subsetneq N_2 \subsetneq N_1 \subsetneq P = N_0$$

such that for some j_{t+1}

$$|(N_t)\pi_{j_{t+1}}| = |N_t| - k_{t+1} + 1,$$

where $k_{t+1} = |N_{t+1}|,\ |(N_{t+1})\pi_{j_{t+1}}| = 1,\ |N_r| = 1$ and $0 \le t \le r-1;\ r > 0;$ if $r = 0$ then $P = N_0,\ |P| = 1$.

Proof. For every non-empty finite subset P with the properties described above we construct a natural ordering $<_P$ of G. In order to describe $<_P$ we have to describe its components: a well order $<_I$ of I and a linear ordering $<_i$ on G_i $\forall i \in I$. We define them as follows.

a) $j_1 <_I j_2 <_I \ldots <_I j_r$ are adjacent elements in the order $<_I$.

b) $\forall i <_I j_1$ it follows that $|(P)\pi_i| = 1$.

c) $<_{j_{t+1}}$ $(0 \leq t \leq r-1)$ is fresh for N_t on $G_{j_{t+1}}$.

d) $<_i$ $(i \notin \{j_1, \dots, j_r\})$ is a linear ordering of G_i.

We will call $<_P$ a special P-order on G. We claim that any special P-order on G is P-fresh if $P \in \Omega = \{$ subsets of G satisfying the conditions of the theorem $\}$.

Let us suppose for contradiction that it is not so. This means that $\emptyset \neq \Omega_1 = \{P \mid P \in \Omega$ such that there exists a special P-order $\sqsubset_P$ which is not P-fresh $\}$. We denote by P an element of Ω_1 of a minimal size. Thus for any $Q \in \Omega$ such that $|Q| < |P|$ we have that any special Q-order is Q-fresh.

Since $P \in \Omega_1$ we have a chain $N_r \subsetneq \dots \subsetneq N_1 \subsetneq P = N_0$. If we have a degenerate case $r = 0$ then $|P| = 1$ and so any natural ordering of G (including a special P-order) is P-fresh. Therefore in this case $P \notin \Omega_1$. Thus our P has size at least 2, which means $r > 0$ and so $N_1 \neq \emptyset$.

If $r = 1$ then any chain for P is of the kind $N_1 \subsetneq P = N_0$, $|N_1| = 1$, $|(P)\pi_{j_1}| = |P|$. We can apply theorem 5.1 (2) and verify that any special P-order satisfies the conditions of the ordering described in that theorem which is fresh for P. Therefore in this case $P \notin \Omega_1$ either. Thus, in fact, our P is such that $|P| \geq 3$ and there exists a chain α of length $r > 1$ (that is to say $|N_1| > 1$).

Next we choose Q to be equal to N_1 from the chain α of P. We have $|N_1| \geq 2$. Clearly $N_1 \in \Omega$. Also $N_1 \subsetneq P = N_0$, which ensures that $|N_1| < |P|$. Therefore we conclude that any special Q-order is Q-fresh.

Now, any special P-order $<_P$ is also a special Q-order $<_Q$ for $Q = N_1$. Indeed, we simply shorten the chain α of P by one link to obtain a chain of $Q = N_1$.

The relevant piece of the chain α of P is

$$N_r \subsetneq \ldots \subsetneq N_2 \subsetneq N_1$$

which after renaming its factors becomes a chain for Q

$$\widehat{N_{r-1}} \subsetneq \ldots \widehat{N_1} \subsetneq Q = \widehat{N_0}.$$

We will call this chain for Q β.

We have our new $\widehat{j_1}, \ldots, \widehat{j_{r-1}}$ which are equal respectively to $j_2, \ldots, j_r$. Thus $\widehat{j_1} <_I \ldots <_I \widehat{j_{r-1}}$ are still adjacent elements in the order $<_I$.

Also for any $i <_I \widehat{j_1}$ we have either $i <_I j_1$ or $i = j_1$. For $i <_I j_1$ we have $|(P)\pi_i| = 1$ and therefore $|(Q)\pi_i| = 1$ since $Q \subset P$. For $i = j_1$ we have $|(Q)\pi_i| = 1$ since $|(N_1)\pi_{j_1}| = 1$ and $Q = N_1$. Thus for any $i <_I \widehat{j_1}$ we have $|(Q)\pi_i| = 1$.

$<_{\widehat{j_{t+1}}}$ $(0 \leq t \leq r-2)$ is fresh for $\widehat{N_t}$ on $G_{\widehat{j_{t+1}}}$ since $<_{\widehat{j_{t+1}}}$ $(0 \leq t \leq r-2)$ stands for $<_{j_{t+2}}$ $(0 \leq t \leq r-2)$ and $\widehat{N_t}$ stands for N_{t+1}. But $<_{j_{t+2}}$ $(0 \leq t \leq r-2)$ is indeed fresh for N_{t+1} because it is the same as to say that $<_{j_{\bar{t}+1}}$ $(1 \leq \bar{t} \leq r-1)$ is fresh for $N_{\bar{t}}$ which is true.

Finally, the condition (d) obviously holds.

Thus we have checked that indeed any special P-order $<_P$ is in fact a special Q-order $<_Q$. Therefore, since every special Q-order is Q-fresh, we conclude that every special P-order $<_P$ is Q-fresh. In particular we have $(\star)$: $\sqsubset_P$ is Q-fresh.

On the other hand since our N_1 is such that $|N_1| \geq 2$ then lemma 5.2 (2) applies. Indeed, we use j_1 as j in the lemma to conclude that $\sqsubset_P$ is not fresh for N_1. We recall that $N_1 = Q$ and so $\sqsubset_P$ is not fresh for Q which contradicts $(\star)$.

This contradiction proves the theorem.

Next we recall that in chapter 4 we introduced the property of groups of being locally fresh in a subset sense and proved (theorem 4.3) that it is equivalent to the property of being fresh. Therefore it is clear by theorem 2.1 of chapter 2 that the property of being locally fresh in a subset sense is inherited by the formation of Cartesian products. However, by the methods exploited above this fact can be proved directly without using the established equivalence of these two notions of being fresh and being locally fresh in a subset sense. We conclude this chapter by demonstrating this direct proof although our elaborations are only interesting by the virtue of being constructive.

We remind the reader the definition of a group being locally fresh in a subset sense which was given in chapter 4.

Definition. *A group G is said to be locally fresh in a subset sense if and only if for any finite non-empty subset P of G there exists an ordering $<^P$ of G which is fresh for P and for any of its non-empty subsets. We say that $<^P$ is an ordering which is P-fresh in a subset sense on G.*

We also recall an obvious observation made in chapter 4 that if an ordering is P-fresh on G in a subset sense for some non-empty finite subset P then it is P-fresh on G. Therefore if a group G is locally fresh in a subset sense then it is locally fresh.

Theorem 5.5. *If $G = Cart_{i \in I} G_i$ where G_i is a group which is locally fresh in a subset sense $\forall i \in I$ then G is a locally fresh group.*

Proof 1. As usual $\forall i \in I$ we denote by π_i a natural homomorphism $G \longrightarrow G_i$ given by $f \mapsto (i)f \ \forall f \in G$.

For every non-empty finite subset P of G we will say that an ordering $<_P$ of G

is an individual P-order in a subset sense if and only if $<_P$ is an individual P-order with $<_i$ being $(P)\pi_i$-fresh in a subset sense $\forall i \in I$.

For any non-empty finite subset P of G such an ordering $<_P$ can be constructed since G_i is locally fresh in a subset sense $\forall i \in I$.

We claim that for any non-empty finite P such an ordering $<_P$ is P-fresh on G. Obviously this implies that G is a locally fresh group, which is exactly what the theorem states.

We assume for contradiction that our claim is wrong. This means that $\emptyset \neq \Omega = \{P \mid \exists <_P \text{ which is not fresh for } P\}$, where $<_P$ is an individual P-order in a subset sense. Among the elements of Ω we choose $\hat{P}$ of a minimal size. Since any linear ordering of G is fresh for a singleton set then $|\hat{P}| \geq 2$. We will denote the individual $\hat{P}$-order in a subset sense which is not fresh for P by $<_{\hat{P}}$.

Now, an ordering $<_{\hat{P}}$ is not fresh for $\hat{P}$ which means that $\exists g \in G$ such that

$$\hat{P}g \subseteq \bigcup_{h <_{\hat{P}} g} \hat{P}h$$

Now, if $\forall i \in I \quad |(\hat{P})\pi_i| = 1$ then we have $|\hat{P}| = 1$, whereas $|\hat{P}| \geq 2 > 1$.

Thus $\exists j \in I$ such that $|(\hat{P})\pi_j| > 1$. Among such $j \in I$ we choose the minimal one with respect to $<_I$. For simplicity we denote it also by j. Now we have $|(\hat{P})\pi_j| > 1, \ |(\hat{P})\pi_k| = 1 \quad \forall k <_I j$.

If $|(\hat{P})\pi_j| = |\hat{P}| = n$ we apply theorem 5.1 (2) to conclude that the ordering described in the theorem is fresh for $\hat{P}$ on G. It follows that $<_{\hat{P}}$ which satisfies the description is also fresh for $\hat{P}$ on G. This contradicts the definition of $\hat{P}$. Thus $1 < |(\hat{P})\pi_j| < n$ which forces $n > 2$.

Let $(\hat{P})\pi_j = \{m_1, \dots, m_s\}, \ 1 < s < n$. Now $\forall k \in \{1, \dots, s\}$ we put $N_k = \{p \in$

$\hat{P} \mid (p)\pi_j = m_k\}$, so the N_k's are the fibres of $\pi_j|_{\hat{P}}$. We arrange matters so that $|N_k| > 1$ $\forall t$ such that $k \leq t < s$ and $t \geq 1$. Now $|\hat{P}| > |N_u| \geq 2$ for all u such that $1 \leq u \leq t$. We now apply lemma 5.2 (2) using $<_{\hat{P}}$ as an ordering $\sqsubset$ to conclude that $<_{\hat{P}}$ is not fresh for N_u where $1 \leq u \leq t$. We focus on N_1.

Now, the ordering $<_{\hat{P}}$ is an individual $\hat{P}$-order on G in a subset sense. This means that $<_i$ is fresh for $(\hat{P})\pi_i$ and for any of its subsets $\forall i \in I$. We have $N_1 \subsetneq P$ and so $(N)\pi_i \subseteq (P)\pi_i \ \ \forall i \in I$, where $N \subseteq N_1$. Therefore we have that $<_i$ is fresh in a subset sense for $(N_1)\pi_i$ on G_i $\forall i \in I$. This means that $<_{\hat{P}}$ is also $<_{N_1}$ - an individual N_1-order in a subset sense on G. Thus $<_{N_1}$ is not fresh for N_1 since as we deduced earlier $<_{\hat{P}}$ is not N_1-fresh.

Therefore we obtain that $N_1 \in \Omega$. On the other hand $|N_1| < |\hat{P}|$ which contradicts the minimality of $|\hat{P}|$.

Thus our assumption was wrong and so an individual P-order in a subset sense is P-fresh for any non-empty finite subset P of G. It implies that G is locally fresh.

The theorem is now proved.

We will now give an alternative proof to the theorem which again does not use the fact that being locally fresh in a subset sense is the same as being fresh.

Proof 2. We claim that for any non-empty finite subset P of G any individual P-order in a subset sense is fresh for P. Let us assume for contradiction that it is not true. Thus we have $\emptyset \neq \Omega = \{P \mid P \subset G,\ |P| < \infty,\ P \neq \emptyset,\ \exists$ an individual P-order in a subset sense which is not P-fresh $\}$.

Among the elements of Ω we choose one of a minimal size. We will denote it by P for simplicity of notation. Clearly, $|P| \geq 2$ since for a singleton set any linear ordering on G is fresh. Next, there is an individual P-order in a subset sense $<_P$ which is not P-fresh, but for any $Q \subset G$, $Q \neq \emptyset$, $|Q| < |P|$ any individual Q-order in a subset sense is Q-fresh. In particular, any of individual Q-orders is fresh for $Q = P - \{p\}$, where p is an arbitrary point of P.

On the other hand, we deduce that $<_P$ is an individual Q-order in a subset sense. Indeed, $\forall i \in I$ we have $<_i$ is fresh for $(P)\pi_i$ and for any of its subsets. Therefore $<_i$ is fresh for $(Q)\pi_i$ and for any of its subsets because $Q \subsetneq P$ and so $(Q)\pi_i \subseteq (P)\pi_i$ and any of the subsets of $(Q)\pi_i$ are also subsets of $(P)\pi_i$. In fact, we have shown that $<_P$ is an individual Q-order in a subset sense whenever $Q \subset P$.

Thus we conclude that $<_P$ is Q-fresh. Therefore $\exists g \in G$ such that

$$Pg \subseteq \bigcup_{h <_P g} Ph.$$

and at the same time

$$Qg \not\subseteq \bigcup_{h <_P g} Qh$$

Choose $q \in Q$ such that $qg \not\subseteq \bigcup_{h <_P g} Qh$ then $qg \in Pg$ so $qg \in (\bigcup_{h <_P g} Ph) - (\bigcup_{h <_P g} Qh)$. Therefore $qg = ph$ for some $h <_P g$.

We denote p by p_1 and q by p_2. So, $p_2 g = p_1 h_1$, $h_1 <_P g$.

Given $p_i \in P$ we obtain p_{i+1} in the same way, using p_i as our original p and finding p_{i+1} as q.

Since P is finite the infinite sequence (p_i) must have a first repeated entry. By omitting initial terms and relabelling we may assume that $p_1, p_2, ..., p_m$ are distinct but $p_{m+1} = p_1$.

Now $p_{i+1} g = p_i h_i$ for some $h_i <_P g \; \forall i \in \{1, \ldots, m\}$. Also $p_1 g = p_{m+1} g = p_m h_m$. We denote $M = \{p_1, ..., p_m\}$. Thus $M \subseteq P$ and $M \in \Omega$.

Indeed, $Mg \subseteq \bigcup_{h <_P g} Mh$ which means that $<_P$ is not fresh for M. Also, $<_P$ is an individual M-order in a subset sense which is always ensured whenever $M \subset P$ as we have shown above.

Now, the minimality of $n = |P|$ forces $n = m$ and so $P = \{p_1, p_2, ..., p_n\}$. Thus we rearranged the points of P in such a way that the following system of equations now holds:

$$p_2 g = p_1 h_1$$

$$\cdots$$

$$p_{k+1} g = p_k h_k \tag{*}$$

$$\cdots$$

$$p_1 g = p_{n+1} g = p_n h_n,$$

where $h_k \in G,\ h_k <_P g\ \forall k \in \{1, \ldots, n\}$ and $p_k \neq p_l$ if $k \neq l$ for $k, l \in \{1, \ldots, n\}$, $p_{n+1} = p_1$.

If $\forall i \in I$ we have $|(P)\pi_i| = 1$ then $|P| = 1$ which is not the case. Therefore $\emptyset \neq J = \{j \in I |\ |(P)\pi_j| > 1\}$. We denote the minimal element of J by j for simplicity of our notations. Clearly j exists since $J \subset I$ which is well ordered.

Next we project (*) onto G_j. We obtain the following system of equations in G_j :

$$(p_2)\pi_j(g)\pi_j = (p_1)\pi_j(h_1)\pi_j$$

$$\dots$$

$$(p_{k+1})\pi_j(g)\pi_j = (p_k)\pi_j(h_k)\pi_j \qquad (**)$$

$$\dots$$

$$(p_1)\pi_j(g)\pi_j = (p_{n+1})\pi_j(g)\pi_j = (p_n)\pi_j(h_n)\pi_j.$$

Here $(p_1)\pi_j, \dots, (p_n)\pi_j$ are not necessarily different since π_j is not necessarily injective. Also $h_k <_P g \; \forall k \in \{1, \dots, n\}$ together with $|(P)\pi_j| > 1$, $|(P)\pi_s| = 1 \; \forall s <_I j$ result in $(h_k)\pi_j \leq_j (g)\pi_j \; \forall k \in \{1, \dots, n\}$, where $\exists \hat{k} \in \{1, \dots, n\}$ such that $(h_{\hat{k}})\pi_j <_j (g)\pi_j$.

Now we can apply lemma 2.1.2 using G_j as G, $<_j$ as $<$ and $(P)\pi_j$ as P in the lemma respectively. We conclude that $<_j$ is not fresh on G_j. Moreover, from the proof of the lemma it is clear that in fact $<$ is not fresh on G for the set T which represents all the different elements of P. Therefore we can deduce that $<_j$ is not fresh on G_j for $(P)\pi_j$. Obviously, therefore $<_j$ is not fresh on G_j for $(P)\pi_j$ in a subset sense either. This contradicts $<_P$ being an individual P-order in a subset sense.

Thus our assumption was wrong and so any individual P-order in a subset sense is fresh for P where P is an arbitrary non-empty finite subset of G. Therefore G is indeed a locally fresh group.

The theorem is now proved.

Finally, we note that similarly to what has been done in this chapter one could speculate on whether theorem 2.2 of chapter 2 can be modified. This is to ask whether or not the property of groups of being locally fresh is preserved by extensions.

CONCLUSION

To conclude the thesis we summarize the work which has been done.

We studied infinite systems of linear equations constructed in a special way using infinite groups. Given a group G we associated each group element with an unknown and constructed an infinite system of equations by encoding certain linear recurrencies. These recurrencies give rise to the concept of templates. In the attempt to study these infinite systems of linear equations we developed a theory of templates and introduced the notion of fresh orderings of groups which arose naturally from this study. The existence of a well fresh ordering of a group ensures that the infinite system of linear equations generated by a single large template has a non-trivial solution.

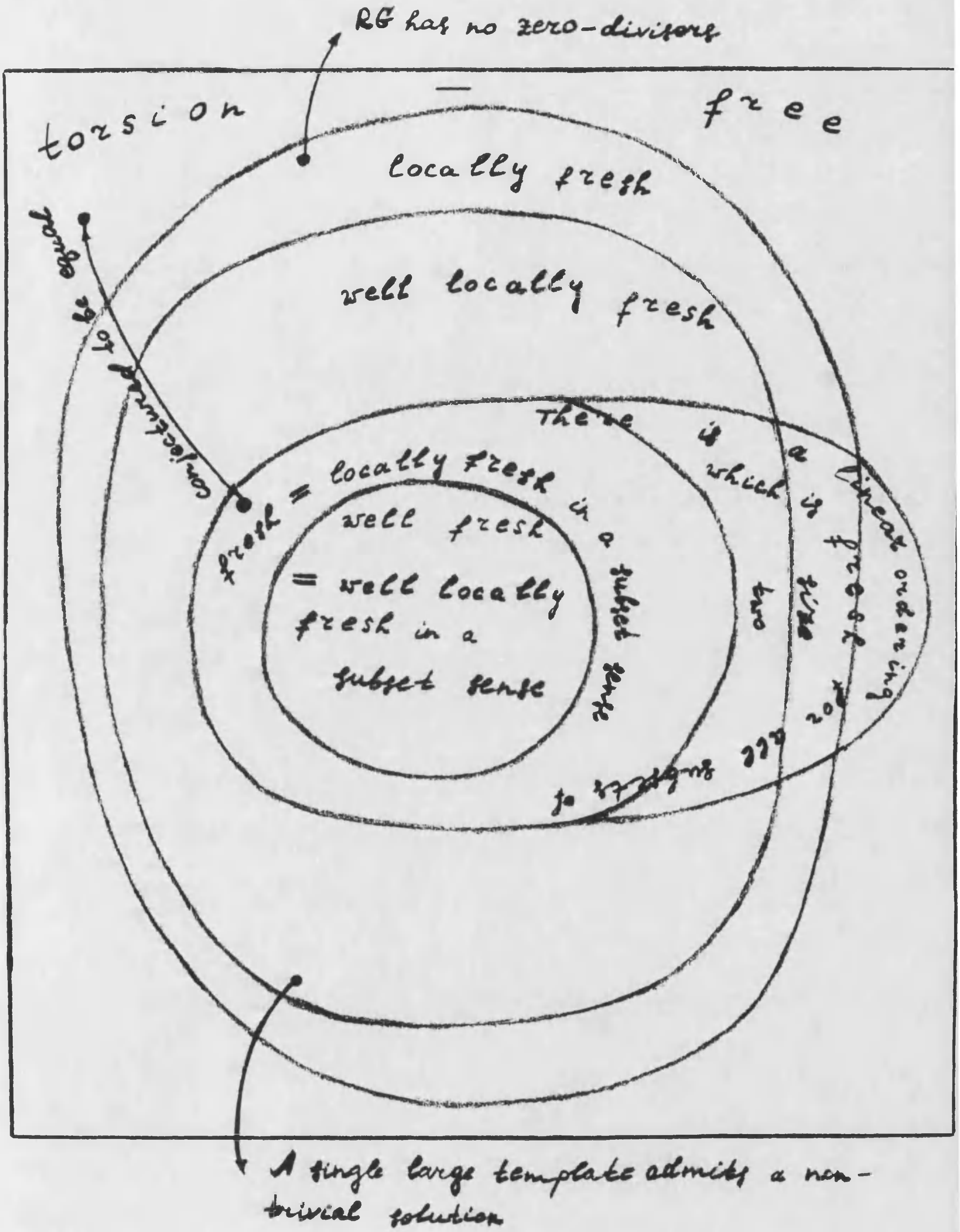
The groups which have fresh orderings we named fresh groups and studied them as a class. We discovered that the class of fresh groups enjoys various closure properties. It is closed under taking subgroups, Cartesian products, extensions and free products. We described various classes of groups which are fresh. They include free groups, free abelian groups, right(left)-orderable groups and they only contain groups which are torsion-free. An attempt was made to understand what theory may be salvaged when modifying the condition of being fresh into that of being locally fresh.

We also proved that the fact that an infinite system of our linear equations has only the trivial solution can be deduced from looking at a finite subset of the system. Thus we justified the success of some computational experiments which were conducted in the past. To obtain this and some other results we used the means of mathematical logic.

Finally, we conjectured that the class of torsion-free groups coincides with the class of fresh groups. This, if true, would give a negative answer to the currently open question on the existence of zero divisors in a group ring RG of a torsion-free group G where R is an integral domain. This is because we have a result stating that RG does not have zero divisors if G is fresh or even locally fresh. However, the class of fresh groups is genuinely different from the class of groups for which this question has been already answered.

In the course of studying fresh groups we modified the definition of a fresh ordering and fresh group respectively and obtained a number of related classes of groups. These are well fresh groups, locally fresh and locally well fresh groups and groups locally fresh in a subset sense. We can draw the following graph which reflects our current knowledge of the way these classes of groups are related.

Note that in this "Venn Diagram" it may so happen that some of the regions contain no groups (for example, perhaps the classes of fresh groups and locally fresh groups coincide). The diagram is given on the next page.


RG has no zero-divisors
torsion — free
locally fresh
well locally fresh
Conjectured to be equal
There is a linear ordering
which is
fresh for all subsets of
two
fresh = locally fresh in a subset sense
well fresh
= well locally fresh in a subset sense
A single large template admits a non-trivial solution

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